MATH 20E Lecture 12 - Tuesday, November 5, 2013: midterm

MATH 20E Lecture 13 - Thursday, November 7, 2013

Gradient fields in the plane

$\vec{F} = M\hat{\imath} + N\hat{\jmath}$ where M, N are functions of x, y.

curl$(\vec{F}) = N_x - M_y$ measures the failure of $\vec{F}$ to be conservative.

Interpretation of curl: for a velocity field, curl = (twice) angular velocity of the rotation component of the motion.

Equivalent properties:

1. $\vec{F}$ is conservative, i.e. if C is a closed loop then $\int_C \vec{F} \cdot d\vec{r} = 0$.
2. Path independence: if C_1, C_2 are curves in the plane that have same endpoints, then
$$\int_{C_1} \nabla f \cdot d\vec{r} = \int_{C_2} \nabla f \cdot d\vec{r}.$$
3. $\vec{F}$ is a gradient field, i.e. there exist a function $f(x, y)$ (called the *potential*) such that $\vec{F} = \nabla f = (f_x, f_y)$.
4. curl $\vec{F} = 0$ and $\vec{F}$ $\vec{F}$ is defined everywhere, or in a simply-connected region (no holes).

Note:

- path independence is equivalent to conservativeness by considering C_1, C_2 with same endpoints, $C = C_1 + C_2^-$ is a closed loop. So (1) $\iff$ (2).
- We have already seen that for a gradient field, we have path independence and conservativeness.

  If C_1, C_2 have same endpoints then $\int_{C_1} \nabla f \cdot d\vec{r} = \int_{C_2} \nabla f \cdot d\vec{r}$ (both equal to $f(P_1) - f(P_0)$ by the theorem). So the line integral $\int_C \nabla f \cdot d\vec{r}$ depends only on the end points, not on the actual trajectory.
  Also, if C is a closed loop then $\int_C \nabla f \cdot d\vec{r} = 0 (= f(P) - f(P))$.

  Hence (3) $\implies$ (1) and (2).
- if we have path independence, then we can get the potential $f(x, y)$ by computing $\int_{(0,0)}^{(x,y)} \vec{F} \cdot d\vec{r}$. Hence (3) $\implies$ (2).
- We know that (3) $\implies$ (4) because $f_{xy} = f_{yx}$.
- Assume $\vec{F}$ is defined and differentiable everywhere (or in a region without holes) and curl $F = 0$. Then Green's theorem says for every closed curve C we have (after possibly changing the orientation), that
$$\int_C \vec{F} \cdot d\vec{r} = \iint_R (\text{curl}\,\vec{F}) dA = \iint_R 0 dA = 0.$$
  Thus (4) $\implies$ (1).

Therefore $\text{curl}(\vec{F}) = N_x - M_y$ measures the failure of $\vec{F}$ to be conservative.

We have seen : $N_x = M_y \overset{*}{\iff} \vec{F}$ is a gradient field $\iff \vec{F}$ is conservative (i.e. $\int_C \vec{F} \cdot d\vec{r} = 0$ for any closed curve C.)

$(*) : \implies$ only holds if $\vec{F}$ is defined everywhere, or in a simply-connected region (no holes).

Interpretation of curl: for a velocity field, curl = (twice) angular velocity of the rotation component of the motion. For a force field, $\text{curl}\,\vec{F}$ = torque exerted on a test mass, measures how $\vec{F}$ imparts rotation motion.

How to find the potential of a gradient field

Note: only try this if $\text{curl}\,\vec{F} = 0$.

Method I: antiderivatives

Example: Find a such that $\vec{F} = (4x^2 + axy, 4x^2 + 3y^2)$ is a gradient field.

$N_x = 8x, M_y = ax$, so $a = 8$.

Example: For the value of a found above, find a potential for $\vec{F}$.

Then $\vec{F} = (4x^2+8xy, 4x^2+3y^2)$ and we are looking for a function $f(x, y)$ such that $f_x = 4x^2+8xy$ and $f_y = 4x^2 + 3y^2$.

Step 1: Since $f_x = 4x^2 + 8xy$ we integrate with respect to x and see that $f(x, y) = \frac{4}{3}x^3 + 4x^2y + g(y)$.

Step 2: differentiate the above with respect to y and compare to N. We want $f_y = 4x^2 + g'(y) = 4x^2 + 3y^2$. So $g'(y) = 3y^2$, and $g(y) = y^3(+cst)$.

Step 3: substitute $g(y)$ into $f(x, y)$ and get $f(x, y) = \frac{4}{3}x^3 + 4x^2y + y^3(+cst)$.

Method II: line integral (FTC backwards)

We know that if C starts at $(0, 0)$ and ends at (x_1, y_1) then $f(x_1, y_1) - f(0, 0) = \int_C \vec{F} \cdot d\vec{r}$.

Here $f(0, 0)$ is just an integration constant (if f is a potential then so is $f + cst$). Can also choose the simplest curve C from $(0, 0)$ to (x_1, y_1). (x1, 0) to (x1, y1) (picture drawn).

Simplest choice: take $C = C_1 + C_2$ with C_1 = portion of x-axis from $(0, 0)$ to $(x_1, 0)$, then C_2 = vertical segment from $(x_1, 0)$ to (x_1, y_1) (picture drawn). *Example:* $\vec{F} = (e^{xy} + xye^{xy})\hat{\mathbf{i}} + x^2e^{xy}\hat{\mathbf{j}}$. (Here it is hard to find the antiderivative of M with respect to x.)

Then
$$\int_C \vec{F} \cdot d\vec{r} = \int_{C_1+C_2} (e^{xy} + xye^{xy})dx + x^2e^{xy}dy.$$

On $C_1 : x = t, y = 0, 0 \le t \le x_1$ so $dx = dt$ and $dy = 0$. Therefore
$$\int_{C_1} \vec{F} \cdot d\vec{r} = \int_0^{x_1} dt = x_1.$$

On $C_2 : x = x_1, y = t, 0 \le t \le y_1$ so $dx = 0$ and $dy = dt$. Therefore
$$\int_{C_2} \vec{F} \cdot d\vec{r} = \int_0^{y_1} x_1^2 e^{x_1 t} dt = \left[x_1 e^{x_1 t}\right]_{t=0}^{t=y_1} = x_1 e^{x_1 y_1} - x_1.$$

Adding up the two integrals we get $f(x_1, y_1) = \int_C \vec{F} \cdot d\vec{r} = x_1 e^{x_1 y_1}$.
So $f(x, y) = xe^{xy}$ is a potential for $\vec{F}$.
Check: $\nabla f = (f_x, f_y) = (e^{xy} + xye^{xy}, x^2 e^{xy}) = \vec{F}$.

Proof of Green's Theorem

Green's Theorem: $\int_C M dx + N dy = \iint_R (N_x - M_y) dA$ where C is a closed curve oriented counter-clockwise enclosing region R of the plane.
Proof: two preliminary remarks:
1) the theorem splits into two identities, $\int_C M dx = -\iint_R M_y dA$ and $\int_C N dy = \int_R N_x dA$.

2) additivity: if theorem is true for R_1 and R_2 then its true for the union $R = R_1 \cup R_2$ (picture drawn): $\int_C = \int_{C_1} + \int_{C_2}$ (the line integrals along inner portions cancel out) and $\iint R = \iint_{R_1} + \iint_{R_2}$.

Main step in the proof: prove $\int_C M dx = -\iint_R M_y dA$ for "vertically simple" regions: $a < x < b, f_1(x) < y < f_2(x)$. (picture drawn). This is enough because we can divide any region into such pieces and use additivity.
LHS: break C into four sides (C_1 lower, C_2 right vertical segment, C_3 upper, C_4 left vertical segment);
$\int_{C_2} M dx = \int_{C_4} M dx = 0$ since $x =$ constant on C_2 and C_4.
On $C_1 : x = x, y = f_1(x), a \le x \le b$ so $\int_{C_1} M(x, y)dx = \int_a^b M(x, f_1(x))dx$
On $C_3 : x = x, y = f_2(x), b \le x \le a$ (because of the orientation) so

$$\int_{C_3} M(x, y)dx = -\int_a^b M(x, f_2(x))dx$$

$$\int_C = \int_{C_1} + \int_{C_3} = \int_a^b (M(x, f_1(x)) - M(x, f_2(x)))\, dx$$

RHS: $\iint_R -M_y dA = -\int_a^b \int_{f_1(x)}^{f_2(x)} M_y dy dx = -\int_a^b (M(x, f_2(x)) - M(x, f_1(x)))\, dx$ (= LHS).
Similarly $\int_C N dy = \iint_R N_x dA$ by subdividing into horizontally simple pieces. This completes the proof.