MATH 20E Lecture 14 - Tuesday, November 12, 2013

Flux in 2D

$\vec{F} = (M, N)$ where M, N are functions of x, y,

The flux of $\vec{F}$ across the plane curve C is by definition by

$$\text{flux} = \int_C \vec{F} \cdot \hat{\mathbf{n}} ds = \int_C M dy - N dx$$

where $\hat{\mathbf{n}}$ = normal vector to C, rotated $90°$ clockwise from $\hat{\mathbf{T}}$. (picture drawn; explained how the coordinate formula comes from the fact that when we rotate vector (a, b) $90°$ clockwise, we get $(b, -a)$.

Physical interpretation: if $\vec{F}$ is a velocity field (e.g. flow of a fluid), flux measures how much matter passes through C per unit time, counting positively what flows towards the right of C, negatively what flows towards the left of C, as seen from the point of view of a point traveling along C.

Look at a small portion of C : locally $\vec{F}$ is constant, what passes through portion of C in unit time is contents of a parallelogram with sides Δs and $\vec{F}$ (picture shown with $\vec{F}$ horizontal, and portion of curve = diagonal line segment). The area of this parallelogram is Δs·height $= \Delta s(\vec{F} \cdot \hat{\mathbf{n}})$. (picture shown rotated with portion of C horizontal, at base of parallelogram). Summing these contributions along all of C, we get that $\int_C \vec{F} \cdot \hat{\mathbf{n}} ds$ is the total flow through C per unit time; counting positively what flows towards the right of C, negatively what flows towards the left of C, as seen from the point of view of a point traveling along C.

Note: in the plane, work and flux have different physical interpretations, but they are both line integrals, so they get setup and evaluated the same way.

Example: C = circle of radius a oriented counterclockwise, $\vec{F} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}}$ (picture shown): along C, $\vec{F} // \hat{\mathbf{n}}$, and $\|\vec{F}\| = a$, so $F \cdot \hat{\mathbf{n}} = a$. So

$$\int_C \vec{F} \cdot \hat{\mathbf{n}} ds = \int_C a ds = a \,\text{length}(C) = 2\pi a^2.$$

Meanwhile, the flux of $\vec{H} = -y\hat{\mathbf{i}} + x\hat{\mathbf{j}}$ across C is zero (field tangent to C). That was a geometric argument. What about the general situation when calculation of the line integral is required? Observe: $d\vec{r} = \hat{\mathbf{T}} ds = (dx, dy)$, and $\hat{\mathbf{n}}$ is $\hat{\mathbf{T}}$ rotated $90°$ clockwise; so $\hat{\mathbf{n}} ds = (dy, -dx)$. So, if $\vec{F} = M\hat{\mathbf{i}} + N\hat{\mathbf{j}}$, then

$$\int_C \vec{F} \cdot \hat{\mathbf{n}} ds = \int_C (M, N) \cdot (dy, -dx) = \int_C M dy - N dx.$$

So we can compute flux using the usual method, by expressing x, y, dx, dy in terms of a parameter variable and substituting (no example given).

Green's theorem for flux (normal form)

If C is a positively oriented (i.e. counterclockwise) closed curve enclosing a region R in the plane, then the flux out of R for the vector field $\vec{F} = M(x, y)\hat{\mathbf{i}} + N(x, y)\hat{\mathbf{j}}$ is

$$\int_C M dy - N dx = \iint_R (M_x + N_y) dA.$$

In vector notation,

$$\int_C \vec{F} \cdot \hat{\mathbf{n}} ds = \iint_R (\operatorname{div} \vec{F}) dA,$$

where $\operatorname{div} \vec{F} = M_x + N_y$.

Proof: $\int_C M dy - N dx = \int_C P dx + Q dy$ with $P = -N$ and $Q = M$. Green's theorem says that $\int_C P dx + Q dy = \iint_R (Q_x - P_y) dA = \iint_R (M_x + N_y) dA$.

Example: in the above example ($\vec{F} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}}$ across circle), $\operatorname{div} \vec{F} = 2$, so flux $= \iint_R 2 dA = 2$ area $(R) = 2\pi a^2$.

If we translate C to a different position (not centered at origin) (picture shown) then direct calculation of flux is harder, but total flux is still $2\pi a^2$.

Physical interpretation: in an incompressible fluid flow, divergence measures source/sink density/rate, i.e. how much fluid is being added to the system per unit area and per unit time.

Flux in 3D

$\vec{F} = P\hat{\mathbf{i}} + Q\hat{\mathbf{j}} + R\hat{\mathbf{k}}$ where P, Q, R are functions of x, y, z. $S =$ surface in space.

If $\vec{F} =$ velocity of a fluid flow, then flux $=$ flow per unit time across surface S.

Cut S into small pieces, then over each small piece: what passes through ΔS in unit time is the contents of a parallelepiped with base ΔS and third side given by $\vec{F}$.

Volume of box $=$ height $\times$ area of base $= (\vec{F} \cdot \hat{\mathbf{n}}) \Delta S$ where $\hat{\mathbf{n}}$ is a unit normal vector to S.

Remark: there are 2 choices for $\hat{\mathbf{n}}$ (choose which way is counted positively $=$ "orientation")

Notation: $d\vec{S} = \hat{\mathbf{n}} dS$ ($d\vec{S}$ is often easier to compute than $\hat{\mathbf{n}}$ and dS separately!).

In 3D, flux of a vector field is the double integral

$$\text{Flux} = \iint_S \vec{F} \cdot \hat{\mathbf{n}} dS = \iint_S \vec{F} \cdot d\vec{S}.$$

Example 1: $\vec{F} = (x, y, z)$ through sphere of radius a centered at 0.

$\hat{\mathbf{n}} = \frac{1}{a}(x, y, z)$ (other choice: $-\frac{1}{a}(x, y, z)$; traditionally choose $\hat{\mathbf{n}}$ pointing out).
$\vec{F} \cdot \hat{\mathbf{n}} = (x, y, z) \cdot \hat{\mathbf{n}} = \frac{1}{a}(x^2 + y^2 + z^2) = a$, so

$$\iint_S F \cdot \hat{\mathbf{n}} dS = \iint_S a dS = a(4\pi a^2).$$

Example 2: Same sphere, $\vec{H} = z\hat{\mathbf{k}}$ Then $\vec{H} \cdot \hat{\mathbf{n}} = \frac{z^2}{a}$ and

$$\iint_S \vec{H} \cdot n dS = \iint_S \frac{z^2}{a} dS$$

Parametrize S by $x = a\cos\theta\sin\phi, y = a\sin\theta\sin\phi, z = a\cos\phi$ with $0 \le \theta \le 2\pi, 0 \le \phi \le \pi$. Then

$$dS = \sqrt{\left(\frac{\partial(y,z)}{\partial(\theta,\phi)}\right)^2 + \left(\frac{\partial(x,z)}{\partial(\theta,\phi)}\right)^2 + \left(\frac{\partial(x,y)}{\partial(\theta,\phi)}\right)^2}\, d\theta d\phi = a^2 \sin\phi\ d\theta d\phi.$$

Flux is given by

$$\iint_S \vec{H}\cdot n dS = \iint_S \frac{z^2}{a} dS = \int_0^{\pi}\int_0^{2\pi} \frac{(a\cos\phi)^2}{a} a^2 \sin\phi\ d\theta d\phi = 2\pi a^3 \int_0^{\pi} \cos^2\phi \sin\phi d\phi = \frac{4\pi a^3}{3}.$$

Setup. Sometimes we have an easy geometric argument, but in general we must compute the surface integral. The setup requires the use of two parameters to describe the surface, and $\vec{F}\cdot\hat{\mathbf{n}}dS$ must be expressed in terms of them. How to do this depends on the type of surface.

1. S = parametric surface with parametrization $\Phi(u,v) = (x(u,v), y(u,v), z(u,v))$ $(u,v) \in R$ some region of the uv-plane.
   normal vector to the surface: $\Phi_u \times \Phi_v$, so unit normal $\hat{\mathbf{n}} = \dfrac{\Phi_u \times \Phi_v}{\|\Phi_u \times \Phi_v\|}$
   surface area element $dS = \|\Phi_u \times \Phi_v\| dudv$

   Hence $d\vec{S} = \hat{\mathbf{n}}dS = (\Phi_u \times \Phi_v) dudv$ and

   $$\boxed{\text{Flux} = \iint_S \vec{F}\cdot d\vec{S} = \iint_R \vec{F}\cdot(\Phi_u \times \Phi_v) dudv.}$$

2. S = graph of a function $g(x,y)$ with x, y in some region R of the xy-plane.

   Then $d\vec{S} = \hat{\mathbf{n}}dS = (-g_x, -g_y, 1)dA$ and

   $$\boxed{\text{Flux} = \iint_S \vec{F}\cdot d\vec{S} = \iint_R \vec{F}\cdot(-g_x, -g_y, 1)dA.}$$

3. S = implicit surface given by equation $f(x,y,z) = 0$.

   Then $d\vec{S} = \hat{\mathbf{n}}dS = \frac{1}{\nabla f\cdot\hat{\mathbf{k}}}\nabla f\, dA$ and

   $$\boxed{\text{Flux} = \iint_S \vec{F}\cdot d\vec{S} = \iint_R \vec{F}\cdot(\nabla f)\frac{1}{\nabla f\cdot\hat{\mathbf{k}}} dA,}$$

   where R is the shadow of S on the xy-plane.

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Example: flux of $\vec{F} = z\hat{\mathbf{k}}$ through S = portion of paraboloid $z = x^2 + y^2$ above the unit disk oriented with normal pointing up (inside the paraboloid); geometrically flux should be > 0 (clicker question). Since S is the graph of the function $f(x, y) = x^2 + y^2$, we have $\hat{\mathbf{n}}dS == (-f_x, -f_y, 1) = (-2x, -2y, 1)$.

$$\iint_S \vec{F} \cdot \hat{\mathbf{n}}dS = \iint_S zdxdy = \iint_S (x^2 + y^2)dxdy = \int_0^{2\pi} \int_0^1 r^2 rdrd\theta = \frac{\pi}{2}.$$

Divergence Theorem (Gauss-Green Theorem)

This is the 3D analogue of Green's theorem for flux.
Divergence theorem: If S is a **closed** surface bounding a region W, with normal pointing **outwards**, and $\vec{F}$ is a vector field defined and differentiable over all of W, then

$$\boxed{\iint_S \vec{F} \cdot d\vec{S} = \iiint_W \operatorname{div} \vec{F} dV.}$$

In coordinates, for $\vec{F} = P(x, y, z)\hat{\mathbf{\imath}} + Q(x, y, z)\hat{\mathbf{\jmath}} + R(x, y, z)\hat{\mathbf{k}}$:

$$\boxed{\iint_S (P, Q, R) \cdot \hat{\mathbf{n}}dS = \iiint_W (P_x + Q_y + R_z)dV}$$

Example: flux of $\vec{H} = z\hat{\mathbf{k}}$ out of sphere of radius a (seen last time): $\operatorname{div} \vec{H} = 0 + 0 + 1 = 1$, so

$$\iint_S \vec{H} \cdot d\vec{S} = \iiint_W 1dV = \operatorname{vol}(W) = \frac{4\pi a^3}{3}.$$

Physical interpretation: $\operatorname{div} \vec{F}$ = source rate = flux generated per unit volume. Imagine an incompressible fluid flow (i.e. a given mass occupies a fixed volume) with velocity $\vec{F}$, then $\iiint_W \operatorname{div} \vec{F} dV = \iint_S \vec{F} \cdot \hat{\mathbf{n}}dS$ says that flux through S is the net amount leaving W per unit time = total amount of sources (minus sinks) in W.

Examples: did exercise 4 from Section 8.4 in the textbook.

Example: take S to be the upper hemisphere $x^2 + y^2 + z^2 = 1$ with $z \geq 0$. Compute the flux of $\vec{F} = 3xy^2\hat{\mathbf{\imath}} + 3x^2y\hat{\mathbf{\jmath}} + z^3\hat{k}$ upward through S.

Flux $= \iint_S \vec{F} \cdot \hat{\mathbf{n}}dS$. In this case $\hat{\mathbf{n}} = (x, y, z)$ and $\vec{F} \cdot \hat{\mathbf{n}} = 6x^2y^2 + z^3$. So flux $= \iint_{\text{unit circle}} 6x^2y^2 + z^4 dS = \iint_{\text{unit circle}} 6x^2y^2 + (1 - x^2 - y^2)^2 dxdy$. Need to parametrize and deal with powers of trig functions, it gets ugly.

We would like to apply Gauss-Green, but cannot do it directly. Instead take S_1 = unit disk in the xy-plane with normal pointing down. Then $S + S_1$ enclose the upper half-ball W of radius 1 and the divergence theorem says that

$$\iint_S \vec{F} \cdot \hat{\mathbf{n}}dS + \iint_{S_1} \vec{F} \cdot \hat{\mathbf{n}}dS = \iiint_W (\operatorname{div} \vec{F})dV.$$

On S_1 the $\hat{\mathbf{n}} = -\hat{\mathbf{k}}$ so $\vec{F} \cdot \hat{\mathbf{n}} = -z^3 = 0$ on S_1. So $\iint_{S_1} \vec{F} \cdot \hat{\mathbf{n}}dS = 0$.

Then $\operatorname{div}\vec{F} = 3(x^2+y^2+z^2)$ and

$$\iint_S \vec{F}\cdot\hat{\mathbf{n}}dS = \iiint_W (\operatorname{div}\vec{F})dV = \int_0^{2\pi}\int_0^{\pi/2}\int_0^1 3\rho^4 \sin\phi d\rho d\phi d\theta = \frac{6\pi}{5}.$$

Del operator ∇

$\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}, \dots\right)$ (symbolic notation!)

For instance, we have seen the notation $\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$, i.e. the gradient.

In 2D the del operator is $\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)$.

For a vector space $\vec{F} = M(x,y)\hat{\mathbf{i}} + N(x,y)\hat{\mathbf{j}}$, we have

$$\nabla\cdot\vec{F} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)\cdot(M,N) = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} = \operatorname{div}\vec{F}.$$

In 3D the del operator is $\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$.

For a vector space $\vec{F} = P(x,y,z)\hat{\mathbf{i}} + Q(x,y,z)\hat{\mathbf{j}} + R(x,y,z)\hat{\mathbf{k}}$, we have

$$\nabla\cdot\vec{F} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\cdot(P,Q,R) = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} = \operatorname{div}\vec{F}.$$

Also, the vector curl of $\vec{F}$ is defined to be

$$\begin{aligned}\nabla\times\vec{F} = \begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ P & Q & R\end{vmatrix} &= \left(\frac{\partial(Q,R)}{\partial(y,z)}\right)\hat{\mathbf{i}} - \left(\frac{\partial(P,R)}{\partial(x,z)}\right)\hat{\mathbf{j}} + \left(\frac{\partial(P,Q)}{\partial(x,y)}\right)\hat{\mathbf{k}}\\ &= \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\hat{\mathbf{i}} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\hat{\mathbf{j}} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\hat{\mathbf{k}}\end{aligned}$$

Note: If $\vec{F} = M(x,y)\hat{\mathbf{i}} + N(x,y)\hat{\mathbf{j}}$ is a plane vector field, we can think of it in space as $\vec{F} = M\hat{\mathbf{i}} + N\hat{\mathbf{j}} + 0\hat{\mathbf{k}}$. In this case, $\nabla\times\vec{F} = (\operatorname{curl}\vec{F})\hat{\mathbf{k}}$.

Example: $\vec{F} = (2xyz, 3x, 5z-2x) \implies \nabla\times\vec{F} = (0, 2xy-2, 3-2xz)$.