## MATH 20E Lecture 16 - Tuesday, November 19, 2013

## Stokes' Theorem

This is another example of FTC in action.
Stokes' Theorem: If $C$ is a closed curve in space, and $S$ any surface bounded by $C$, then

$$
\int_{C} \vec{F} \cdot d \vec{r}=\iint_{S}(\nabla \times \vec{F}) \cdot \hat{\mathbf{n}} d S
$$

Orientation: compatibility of an orientation of $C$ with an orientation of $S$ (changing orientation changes sign on both sides of Stokes).
Rule: if I walk along $C$ in positive direction, with $S$ to my left, then $\hat{\mathbf{n}}$ is pointing up. (Various examples shown.)
Another formulation (right-hand rule): if thumb points along $C$ (1-D object), index finger towards $S$ (2-D object), then middle finger points along $\hat{\mathbf{n}}$ (3-D object).
(Various examples shown.)

## Why is Stokes' Theorem true?

Stokes' Theorem: If $C$ is a closed curve in space, and $S$ any surface bounded by $C$ with compatible orientation, then

$$
\int_{C} \vec{F} \cdot d \vec{r}=\iint_{S}(\nabla \times \vec{F}) \cdot \hat{\mathbf{n}} d S
$$

"Proof" of Stokes: 1) if $C$ and $S$ are in the $x y$-plane then the statement follows from Green.
2) if $C$ and $S$ are in an arbitrary plane: this also reduces to Green in the given plane. Green/Stokes works in any plane because of geometric invariance of work, curl and flux under rotations of space. They can be defined in purely geometric terms so as not to depend on the coordinate system $(x, y, z)$; equivalently, we can choose coordinates $(u, v, w)$ adapted to the given plane, and work with those coordinates, the expressions of work, curl, flux will be the familiar ones replacing $x, y, z$ with $u, v, w$.
3) in general, we can decompose $S$ into small pieces, each piece is nearly flat (slanted plane); on each piece we have approximately work = flux by Greens theorem. When adding pieces, the line integrals over the inner boundaries cancel each other and we get the line integral over $C$; the flux integrals add up to flux through $S$.

Attention! A special case of Stokes' Theorem is when $S=$ surface in space that has no boundary, i.e. it is closed (e.g. sphere, torus). Then Stokes' Theorem tells us that $\iint_{S} \nabla \times \vec{F} d \vec{S}=0$ (the flux of the vector curl across $S$ is 0 ). This holds for any $\vec{F}=P \hat{\mathbf{1}}+Q \hat{\mathbf{j}}+R \hat{\mathbf{k}}$ where $P, Q, R$ are functions of $x, y, z$ defined and differentiable everywhere on $S$.

## Stokes and surface independence

Remark: In Stokes theorem we are free to choose any surface $S$ bounded by $C$ ! (e.g. if $C=$ circle, $S$ could be a disk, a hemisphere, a cone, ...) In Stokes we can choose any surface $S$ bounded by $C$ :
so if a same $C$ bounds two surfaces $S_{1}, S_{2}$, then $\int_{C} \vec{F} \cdot d \vec{r}=\iint_{S_{1}}(\nabla \times \vec{F}) \cdot \hat{\mathbf{n}} d S=\iint_{S_{2}}(\nabla \times \vec{F}) \cdot \hat{\mathbf{n}} d S$ ? Can we prove directly that the two flux integrals are equal?

Answer: change orientation of $S_{2}$, then $S=S_{1}-S_{2}$ is a closed surface with $\hat{\mathbf{n}}$ pointing outwards; so we can apply the divergence theorem: $\iint_{S}(\nabla \times \vec{F}) \cdot \hat{\mathbf{n}} d S=\iiint_{W} \operatorname{div}(\nabla \times \vec{F}) d V$. But $\operatorname{div}(\nabla \times \vec{F})=0$ always. (Checked by calculating in terms of components of $\vec{F}$.)

Example: verify Stokes for $F=z \hat{\mathbf{1}}+x \hat{\mathbf{j}}+y \hat{\mathbf{k}}, C=$ unit circle in $x y$-plane (counterclockwise), $S=$ piece of paraboloid $z=1-x^{2}-y^{2}$.

Direct calculation: $x=\cos t, y=\sin t, z=0$, so

$$
\int_{C} \vec{F} \cdot d \vec{r}=\int_{C} z d x+x d y+y d z=\int_{C} x d y=\int_{0}^{2 \pi} \cos ^{2} t d t=\int_{0}^{2 \pi} \frac{1+\cos 2 t}{2} d t=\pi .
$$

By Stokes: $\nabla \times \vec{F}=(1,1,1)$, and $\hat{\mathbf{n}} d S=\left(-g_{x},-g_{y}, 1\right) d x d y$ for $g(x, y)=1-x^{2}-y^{2}$. So $\hat{\mathbf{n}} d S=(2 x, 2 y, 1) d x d y$ and

$$
\begin{aligned}
& \iint_{S}(\nabla \times \vec{F}) \cdot \hat{\mathbf{n}} d S=\iint_{\text {unit disk }}(1,1,1) \cdot(2 x, 2 y, 1) d x d y=\iint(2 x+2 y+1) d x d y=\iint 1 d x d y=\text { area }(\text { disk })=\pi . \\
& \quad\left(\iint x d x d y=0 \text { by symmetry and similarly for } y\right) .
\end{aligned}
$$

## MATH 20E Lecture 17 - Thursday, November 21, 2013

Example 1: Let $\vec{F}=-2 x z \hat{\mathbf{1}}+y^{2} \hat{\mathbf{k}}$.
a) Calculate $\nabla \times \vec{F}$.
b) Show that $\iint_{R}(\nabla \times \vec{F}) \cdot \hat{\mathbf{n}} d S=0$ for any portion $R$ of the unit sphere $x^{2}+y^{2}+z^{2}=1$. (take the normal vector fi pointing outward)
c) Show that $\int_{C} \vec{F} \cdot d \vec{r}=0$ for any simple closed curve $C$ on the unit sphere $x^{2}+y^{2}+z^{2}=1$.

Solution: a) $\nabla \times \vec{F}=\left|\begin{array}{ccc}\hat{\mathbf{1}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -2 x z & 0 & y^{2}\end{array}\right|=(2 y,-2 x, 0)$.
b) On the unit sphere, $\hat{\mathbf{n}}=(x, y, z)$ so $(\nabla \times \vec{F}) \cdot \hat{\mathbf{n}}=2 y x-2 x y=0$. Therefore $\iint_{R}(\nabla \times \vec{F}) \cdot \hat{\mathbf{n}} d S=$ 0.
c) By Stokes' Theorem $\int_{C} \vec{F} \cdot d \vec{r}=\iint_{R}(\nabla \times \vec{F}) \cdot \hat{\mathbf{n}} d S$ where $R$ is the region delimited by $C$ on the unit sphere. Using the result of b), we get $\int_{C} \vec{F} \cdot d \vec{r}=\iint_{R}(\nabla \times \vec{F}) \cdot \hat{\mathbf{n}} d S=0$.

Example 2: Let $C$ be a simple closed plane curve going counterclockwise around a region $R$. Let $M=M(x, y)$. Express $\int_{C} M d x$ as a double integral over $R$.

Solution: this is an application to Green's theorem. Get $\int_{C} M d x=\iint_{R}-M_{y} d A$.
Example 3: Let $S$ be the part of the spherical surface $x^{2}+y^{2}+z^{2}=2$ lying in $z>1$. Orient $S$ upwards and give its bounding circle, $C$, lying in $z=1$ the compatible orientation.
a) Parametrize $C$ and use the parametrization to evaluate the line integral

$$
I=\int_{C} x z d x+y d y+y d z
$$

b) Compute the vector curl of the vector field $\vec{F}=x z \hat{\mathbf{1}}+y \hat{\mathbf{j}}+y \hat{\mathbf{k}}$.
c) Write down a flux integral through $S$ which can be computed using the value of $I$.

Solution: a) $z=1$ and $x^{2}+y^{2}+z^{2}=1$, so $x^{2}+y^{2}=1$. Therefore $C$ is the circle of radius 1 in the $z=1$ plane. Compatible orientation: counterclockwise.

Parametrization: $x=\cos t, y=\sin t, z=1$ Therefore $d x=-\sin t d t, d y=\cos t d t, d z=0$.

$$
I=\int_{C} x z d x+y d y+y d z=\int_{0}^{2 \pi}(-\cos t \sin t+\cos t \sin t) d t=0
$$

b)

$$
\nabla \times \vec{F}=\left|\begin{array}{ccc}
\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
x z & y & y
\end{array}\right|=\hat{\mathbf{i}}+x \hat{\mathbf{j}}
$$

c) By Stokes' Theorem

$$
\int_{C} \vec{F} \cdot d \vec{r}=\iint_{S}(\nabla \times \vec{F}) \cdot \hat{\mathbf{n}} d S
$$

$\hat{\mathbf{n}}$ is the normal pointing upwards, so $\hat{\mathbf{n}}=\frac{(x, y, z)}{\sqrt{2}}$ on the upper hemisphere of radius $\sqrt{2}$. Thus

$$
I=\int_{C} \vec{F} \cdot d \vec{r}=\iint_{S}(1, x, 0) \cdot \frac{(x, y, z)}{\sqrt{2}} d S=\iint_{S} \frac{x+x y}{\sqrt{2}} d S
$$

## Conservative vector fields

Example: $\vec{F}=(y z, x z, x y) . C: x=t^{3}, y=t^{2}, z=t, 0 \leq t \leq 1$. Then $d x=3 t^{2} d t, d y=2 t d t, d z=d t$ and substitute:

$$
\int_{C} \vec{F} \cdot d \vec{r}=\int_{C} y z d x+x z d y+x y d z=\int_{0}^{1} t^{3}\left(3 t^{2} d t\right)+t^{4}(2 t d t)+t^{5} d t=\int_{0}^{1} 6 t^{5} d t=1
$$

Same $\vec{F}$, curve $C^{\prime}=$ segments from $(0,0,0)$ to $(1,0,0)$ to $(1,1,0)$ to $(1,1,1)$. In the xy-plane, $z=0 \Longrightarrow \vec{F}=x y \hat{\mathbf{k}}$, so $\vec{F} \cdot d \vec{r}=0$, no work on either $C_{1}$ or $C_{2}$. For the last segment, $x=y=1, d x=d y=0$, so $\vec{F}=(z, z, 1)$ and $d \vec{r}=(0,0, d z)$. We get $\int_{C_{3}} \vec{F} \cdot d \vec{r}=\int_{0}^{1} 1 d z=1$.
Both give the same answer because $\vec{F}$ is conservative, in fact $\vec{F}=\nabla(x y z)$.
Recall the fundamental theorem of calculus for line integrals:

$$
\int_{P_{0}}^{P_{1}} \nabla f \cdot d \vec{r}=f\left(P_{1}\right)-f(P 0)
$$

## Gradient fields

$\vec{F}=(P, Q, R) \stackrel{?}{=}\left(f_{x}, f_{y}, f_{z}\right)$ Then $f_{x y}=f_{y x}, f_{x z}=f_{z x}, f_{y z}=f_{z y}$, so $P_{y}=Q_{x}, P_{z}=R_{x}, Q_{z}=$ $R_{y} \Longleftrightarrow \nabla \times \vec{F}=\left(R_{y}-Q_{z}, P_{z}-R_{x}, Q_{x}-P_{y}\right)=(0,0,0)$.

Criterion: $\vec{F}$ is a gradient field if and only if $\nabla \times \vec{F}=0$ and $\vec{F}$ is defined in whole space or "simply connected" region of the space.

Definition: a region $W$ is simply connected if every closed loop $C$ inside $W$ bounds some surface $S$ inside $W$.

Examples: the complement of the $z$-axis is not simply connected (shown by considering a loop encircling the $z$-axis); the complement of the origin is simply connected. A ball is simply connected. A sphere is simply connected; torus is not (in fact it has two independent loops that don't bound). (Pictures drawn)

In the plane: a region $R$ is simply connected if for every closed loop $C$ inside $R$ the interior bounded by $C$ is contained in $R$. For instance, the plane with the origin removed is not simply connected. Shown using the unit circle. (Picture drawn.)

Proof of criterion: assume $\vec{F}$ defined in simply connected region $W$ and with $\nabla \times \vec{F}=0$. Consider two curves $C_{1}$ and $C_{2}$ with same end points. Then $C=C_{1}-C_{2}$ is a closed curve so bounds some $S \subset W$. Stokes' Theorem tells us that

$$
\int_{C_{1}} \vec{F} \cdot d \vec{r}-\int_{C_{2}} \vec{F} \cdot d \vec{r}=\int_{C} \vec{F} \cdot d \vec{r}=\iint_{S}(\nabla \times \vec{F}) \cdot \hat{\mathbf{n}} d S=0 .
$$

Thus we get path independence $\Longrightarrow \vec{F}$ conservative $\Longrightarrow$ can find potential

$$
f(x, y, z)=\int_{A}^{(x, y, z)} \vec{F} \cdot d \vec{r} .
$$

Here $A$ is some point in $W$.
Example: (a) for which $a, b$ is $\vec{F}=\left(a x y, x^{2}+z^{3}, b y z^{2}-4 z^{3}\right)$ a gradient field? (b) For the $a, b$ found above, find a potential for $\vec{F}$.
(a) $P_{y}=a x=2 x=Q_{x}$ so $a=2 ; P_{z}=0=0=R_{x} ; Q_{z}=3 z^{2}=b z^{2}=R_{y}$ so $b=3$.
(b) Method I: antiderivatives.

Want to find $f(x, y, z)$ such that $\nabla f=\vec{F}$. That is, we want $f_{x}=P=2 x y, f_{y}=Q=x^{2}+z^{3}, f_{z}=$ $R=3 y z^{2}-4 z^{3}$.

Integrate w.r.t. $x$ the relation

$$
f_{x}=2 x y \xrightarrow{\int d x} f=x^{2} y+g(y, z)
$$

Take the partial derivative w.r.t. $y$ of both sides of this equation and compare it with the relation $f_{y}=Q$. Get

$$
f=x^{2} y+g(y, z) \xrightarrow{\partial_{y}} f_{y}=x^{2}+g_{y}=x^{2}+z^{3}, \text { so } g_{y}=z^{3} \xrightarrow{\int d y} g=y z^{3}+h(z) .
$$

Thus $f=x^{2} y+y z^{3}+h(z)$ and we take the partial derivative w.r.t. $z$ of both sides of this equation and compare it with the relation $f_{z}=R$. Get

$$
f=x^{2} y+y z^{3}+h(z) \xrightarrow{\partial_{z}} f_{z}=3 y z^{2}+h^{\prime}(z)=3 y z^{2}-4 z^{3}, \text { so } h^{\prime}(z)=-4 z^{3} \xrightarrow{\int d z} h=-z^{4}(+C) .
$$

We obtain $f(x, y, z)=x^{2} y+y z^{3}-z^{4}(+C)$.
Check: $\nabla f=\left(2 x y, x^{2}+z^{3}, 3 y z^{2}-4 z^{3}\right)=\vec{F}$.

