MATH 20E Lecture 18 - Tuesday, November 26, 2013

Gradient fields - continued

Example: Last time we have seen that $\vec{F} = (2xy, x^2 + z^3, 3yz^2 - 4z^3)$ is a gradient field. To find its potential we can use two systematic methods:
Method I: antiderivatives. (done last time)

Method II: use path-independence

$f(x_1, y_1, z_1) = \int_C F \cdot d\vec{r} = \int_C 2xydx + (x^2 + z^3)dy + (3yz^2 - 4z^3)dz$ where C is a curve of your choice going from $(0, 0, 0)$ to (x_1, y_1, z_1).

Use a curve that gives an easy computation, e.g. 3 segments parallel to axes. Namely, take $C = C_1 + C_2 + C_3$ where C_1 = segment from $(0, 0, 0)$ to $(x_1, 0, 0)$; C_2 = segment from $(x_1, 0, 0)$ to $(x_1, y_1, 0)$; C_3 = segment from $(x_1, y_1, 0)$ to (x_1, y_1, z_1).

On $C_1 : x = t, y = 0, z = 0, 0 \leq t \leq x_1; dx = dt, dy = dz = 0 \implies \int_{C_1} F \cdot d\vec{r} = \int_{C_1} 2xydx + (x^2 + z^3)dy + (3yz^2 - 4z^3)dz = \int_{C_1} 0 = 0$.

On $C_2 : x = x_1, y = t, z = 0, 0 \leq t \leq y_1; dx = 0, dy = dt, dz = 0 \implies \int_{C_2} F \cdot d\vec{r} = \int_{C_2} 2xydx + (x^2 + z^3)dy + (3yz^2 - 4z^3)dz = \int_0^{y_1} x_1^2 dy = x_1^2 y_1$.

On $C_3 : x = x_1, y = y_1, z = t, 0 \leq t \leq z_1; dx = dy = 0, dz = dt \implies \int_{C_3} F \cdot d\vec{r} = \int_{C_3} 2xydx + (x^2 + z^3)dy + (3yz^2 - 4z^3)dz = \int_0^{z_1} (3y_1t^2 - 4t^3)dt = y_1 z_1^3 - z_1^4$.
So $f(x_1, y_1, z_1) = x_1^2 y_1 + y_1 z_1^3 - z_1^4$.
Check: $f(x, y, z) = x^2y + yz^3 - z^4 \implies \nabla f = (2xy, x^2 + z^3, 3yz^2 - 4z^3) = \vec{F}$. ☺

Applications of div and vector curl to physics

Recall: vector curl of velocity field = 2·angular velocity vector (of the rotation component of motion). E.g., for uniform rotation about z-axis, $\vec{v} = \omega(-y\hat{\mathbf{i}} + x\hat{\mathbf{j}})$ and $\nabla \times \vec{v} = 2\omega\hat{\mathbf{k}}$.
Vector curl singles out the rotation component of motion (while div singles out the stretching component).

Interpretation of vector curl for force fields

If we have a solid in a force field (or rather an acceleration field!) $\vec{F}$ such that the force exerted on Δm at (x, y, z) is $F(x, y, z)\Delta m$: recall the torque of the force about the origin is defined as $\tau = \vec{r} \times \vec{F}$ and measures how $\vec{F}$ imparts rotation motion.

For translation motion: $\dfrac{\text{Force}}{\text{mass}} = \text{acceleration} = \dfrac{d}{dt}(\text{velocity})$.

For rotation effects: $\dfrac{\text{Torque}}{\text{moment of inertia}} = \text{angular acceleration} = \dfrac{d}{dt}(\text{angular velocity})$.

Hence: vector curl of $\dfrac{\text{Force}}{\text{mass}} = 2 \cdot \dfrac{\text{Torque}}{\text{moment of inertia}}$.

Consequence: if $\vec{F}$ derives from a potential, then $\nabla \times \vec{F} = \nabla \times (\nabla f) = 0$, so $\vec{F}$ does not induce any rotation motion. E.g., gravitational attraction by itself does not affect Earth's rotation. (not strictly true: actually Earth is deformable; similarly, friction and tidal effects due to Earth's gravitational attraction explain why the Moon's rotation and revolution around Earth are synchronous).

Div and vector curl of electrical field – part of Maxwells equations for electromagnetic fields.

Gauss-Coulomb law: $\operatorname{div} \vec{E} = \frac{\rho}{\epsilon_0}$ (ρ = charge density and ϵ_0 = physical constant).

By divergence theorem, can reformulate as: $\displaystyle\iint_S \vec{E} \cdot \hat{\mathbf{n}} dS = \iiint_W (\operatorname{div} \vec{E}) dV = \frac{Q}{\varepsilon_0}$, where Q = total charge inside the closed surface S.
This formula tells how charges influence the electric field; e.g., it governs the relation between voltage between the two plates of a capacitor and its electric charge.

Faraday law: $\nabla \times \vec{E} = -\dfrac{\partial \vec{B}}{\partial t}$ ($\vec{B}$ is a magnetic field) So in presence of a varying magnetic field, $\vec{E}$ is no longer conservative: if we have a closed loop of wire, we get a non-zero voltage ("induction" effect). By Stokes,

$$\int_C \vec{E} \cdot d\vec{r} = -\frac{d}{dt} \iint_S \vec{B} \cdot \hat{\mathbf{n}} dS.$$

This principle is used e.g. in transformers in power adapters: AC runs through a wire looped around a cylinder, which creates an alternating magnetic field; the flux of this magnetic field through another output wire loop creates an output voltage between its ends.

There are two more Maxwell equations, governing div and vector curl of $\vec{B}$:

$$\nabla \cdot \vec{B} = 0$$

and

$$\nabla \times \vec{B} = 0 = \mu_0 \vec{J} + \epsilon_0 \mu_0 \frac{\partial \vec{E}}{\partial t} \text{ (where } \vec{J} = \text{current density).}$$