HOMEWORK 4

DUE 4 FEBRUARY 2015

SHOW ALL YOUR WORK.

1. Let d be a nonzero integer (could be negative!).

(a) Prove that
$$\mathbb{Z}[\sqrt{d}] = \{a + b\sqrt{d}\,; a, b \in \mathbb{Z}\}$$
is a ring and that
$$\mathbb{Q}(\sqrt{d}) = \{x + y\sqrt{d}\,; x, y \in \mathbb{Q}\}$$
is a field.

(b) Show that if d is a perfect square, then $\mathbb{Z}[\sqrt{d}] = \mathbb{Z}$ and $\mathbb{Q}(\sqrt{d}) = \mathbb{Q}$.

(c) Furthermore, if $d = d_1 d_2^2$ then $\mathbb{Q}(\sqrt{d}) = \mathbb{Q}(\sqrt{d_1})$.

(d) If d is not a perfect square, show that for each element $\alpha \in \mathbb{Q}(\sqrt{d})$ there exist unique $x, y \in \mathbb{Q}$ such that $\alpha = x + y\sqrt{d}$.

2. Find

(a) the inverse of $5 + 4\sqrt{3}$ in $\mathbb{Q}(\sqrt{3})$;

(b) $\dfrac{7+4i}{9+16i}$ in $\mathbb{Q}(i)$;

(c) $\dfrac{9+4\sqrt{5}}{3-2\sqrt{5}}$ in $\mathbb{Q}(\sqrt{5})$;

(d) $\dfrac{9+4\sqrt{-5}}{3-2\sqrt{-5}}$ in $\mathbb{Q}(\sqrt{-5})$.

3. Find the prime factorization of the following integers in $\mathbb{Z}[i]$.

(a) 23
(b) 13
(c) 17
(d) 296
(e) 415

4. Find the prime factorization of the following gaussian integers in $\mathbb{Z}[i]$.
(a) $2 + 12i$
(b) $3 + 4i$
(c) $7 + 3i$
(d) $10 + 9i$
(e) $10 + 91i$

5. Can 35 be written as the sum of two squares? How about 45? How about 245?

6. Find an integer that can be written as sum of two squares in 3 different ways.

7. Find an integer that can be written as sum of two squares in 4 different ways.

8. Find an integer that can be written as sum of two squares in 5 different ways.