HOMEWORK 6

DUE 18 FEBRUARY 2015

SHOW ALL YOUR WORK.

1. Exercise 9.12 from the notes.

2. Suppose $d = a^2$ is a perfect square. Find all the integer solutions of the Fermat-Pell equation
$$x^2 - dy^2 = 1.$$

3. The number
$$\gamma = \frac{1+\sqrt{5}}{2}$$
is called the *golden ratio.* For each $0 \leq y \leq 20$ find the integer x making $|x - y\gamma|$ as small as possible. Which rational number x/y with $1 \leq y \leq 20$ most closely approximates γ? Feel free to use a computer or calculator.

4. Find the smallest positive integer solution to the following equations or show that no positive integer solutions exist.

(a) $x^2 - 31y^2 = 1$

(b) $x^2 - 30y^2 = -1$

(c) $x^2 - 29y^2 = 1.$

5. (a) Find *all* integer solutions, or prove that no such solutions exist, to the equation
$$x^2 - 7y^2 = -1.$$

(b) Find *all* integer solutions, or prove that no such solutions exist, to the equation
$$x^2 - 7y^2 = 1.$$

6. Let d be an integer that is not a perfect square (could be negative).

(a) For any $n \in \mathbb{Z}_{>0}$ and any $x, y \in \mathbb{Z}$ show that there exist $A, B \in \mathbb{Z}$ such that $(x + y\sqrt{d})^n = A + B\sqrt{d}$ and $(x - y\sqrt{d})^n = A - B\sqrt{d}$.

(b) For any $n \in \mathbb{Z}_{>0}$ and any $x, y \in \mathbb{Q}$ show that there exist $A, B \in \mathbb{Q}$ such that $(x + y\sqrt{d})^n = A + B\sqrt{d}$ and $(x - y\sqrt{d})^n = A - B\sqrt{d}$.

(c) For any $n \in \mathbb{Z}$ and any $x, y \in \mathbb{Q}$ not both zero show that there exist $A, B \in \mathbb{Q}$ such that $(x + y\sqrt{d})^n = A + B\sqrt{d}$ and $(x - y\sqrt{d})^n = A - B\sqrt{d}$.