HOMEWORK 7

DUE 4 MARCH 2015

SHOW ALL YOUR WORK.

1. For the following pairs of polynomials $f(X), g(X)$ indicate if $f(X) \mid g(X)$ in $\mathbb{Q}[X]$ and $\mathbb{Z}[X]$.
(a) $f(X) = 2X + 1$ and $g(X) = 6X + 3$;
(b) $f(X) = 4X + 2$ and $g(X) = 6X + 3$;
(c) $f(X) = X^2 - 2$ and $g(X) = X^6 - 6X^4 + 12X^2 - 8$.

2. Let $\alpha \in \mathbb{C}$. Prove that there exists at most one nonzero polynomial $m(X) \in \mathbb{Q}[X]$ with the following properties
- $m(\alpha) = 0$;
- $m(X)$ is monic (i.e. the coefficient of the highest power of X is 1);
- if $f(X) \in \mathbb{Q}[X]$ nonzero polynomial with $f(\alpha) = 0$, then $\deg m \leq \deg f$.

If such an m exists, it is called *the minimal polynomial* of α over $\mathbb{Q}$.

3. Prove that if $\alpha \in \mathbb{C}$ is a root of the polynomial $f(X) \in \mathbb{Q}[X]$ then the minimal polynomial of α over $\mathbb{Q}$ divides f in $\mathbb{Q}[X]$.

4. Prove that the minimal polynomial of $\alpha \in \mathbb{C}$ over $\mathbb{Q}$ has degree 1 if and only if $\alpha \in \mathbb{Q}$.

5. If $\alpha \in \mathbb{C}$ is an algebraic number over $\mathbb{Q}$ (i.e. it is the root of some polynomial with rational coefficients), prove that there exists a unique nonzero polynomial $f(X) = a_nX^n + \cdots + a_0 \in \mathbb{Z}[X]$ with the following properties
- $f(\alpha) = 0$;
- $\gcd(a_0, \ldots, a_n) = 1$;
- $a_n > 0$;
- if $g(X) \in \mathbb{Z}[X]$ nonzero polynomial with $g(\alpha) = 0$, then $\deg f \leq \deg g$.

6. If α is a quadratic irrational prove that its minimal polynomial over $\mathbb{Q}$ is quadratic and that there exists a unique polynomial $f(X) = aX^2 + bX + c \in \mathbb{Z}[X]$ with $\gcd(a, b, c) = 1$ and $a > 0$ such that $f(\alpha) = 0$.

7. If $\alpha = \sqrt{\frac{m}{n}}$ is irrational and $f(X) = aX^2 + bX + c \in \mathbb{Z}[X]$ has the property that $f(\alpha) = 0$, prove that $b = 0$ and $am + cn = 0$.