HOMEWORK 6

DUE 25 MAY 2015

SHOW ALL YOUR WORK.

Part I. From the textbook (i.e. Leveque)

Section 6.5: 1

Section 6.8: 4

Part II.

1. Prove that

$$\sum_{p\leq x}\frac{1}{p} > \frac{1}{2}\log\log x.$$

Conclude that $\displaystyle\sum_{p}\frac{1}{p}$ diverges.

2. Prove that

(a) If $F(n)=\displaystyle\sum_{d|n}f(d)$, then

$$\sum_{m=1}^{n}F(m)=\sum_{m=1}^{n}\left\lfloor\frac{n}{m}\right\rfloor f(m).$$

(b)

$$\sum_{m=1}^{n}\tau(m)=n\log n+O(n)$$

Hint: see Theorem 6.29 in the textbook.

3. Show that for any complex number $a_1,\dots,a_n$ and $b_1,\dots b_n$, we have

$$\left|\sum_{k=1}^{n}a_kb_k\right|^2\leq\left(\sum_{k=1}^{n}|a_k|^2\right)\left(\sum_{k=1}^{n}|b_k|^2\right)$$

This is called the Cauchy-Schwarz inequality.

4. Fix $k \geq 2$ and define

$$c_k(n) = \begin{cases} 1 & \text{if } n \text{ is } k\text{th power free;} \\ 0 & \text{otherwise.} \end{cases}$$

Show that

$$\sum_{n=1}^{\infty} \frac{c_k(n)}{n^s} = \frac{\zeta(s)}{\zeta(ks)}.$$

Deduce that, in particular,

$$\sum_{n=1}^{\infty} \frac{|\mu(n)|}{n^s} = \frac{\zeta(s)}{\zeta(2s)}.$$

5. Prove that

$$\sum_{m=1}^{n} \sigma(m) = \frac{\pi^2 n^2}{12} + O(n \log n).$$

Hint: See Problem 1 in Section 6.11 of the textbook.

6. **(Extra credit)** Show that

$$\sum_{m=1}^{n} \tau(m) = n \log n + (2\gamma - 1)n + O(\sqrt{n}),$$

where γ is Euler's constant.
Hint: See Theorem 6.30 in the textbook.