MATH 20C – MIDTERM 1
ANSWERS TO PRACTICE PROBLEMS

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This are just answers, not completely written out solutions!

Problem 1: (a) $\dfrac{\sqrt{6}}{2}$

(b) $x + y + 2z = 3$.

(c) $(0, 1, 1)$.

Problem 2: (a) $\overrightarrow{QP} = \mathbf{\hat{i}} - 2\mathbf{\hat{j}}, \quad \overrightarrow{QR} = -2\mathbf{\hat{j}} + 3\mathbf{\hat{k}}$

(b) $\dfrac{4}{\sqrt{65}}$

(c) $\dfrac{9}{\sqrt{130}}$

Problem 3: (a) $7/2$

(b) $6x + 3y + 2z = 11$

(c) Line has direction $\langle 1, 0, -3\rangle$ and $\vec{N} \cdot \langle 1, 0, -3\rangle = 0$. Therefore line $\perp \vec{N}$ and so it is parallel to the plane.

Problem 4: $\vec{L}(u) = \langle -1, 1, 1 + u\rangle$

Problem 5: $\pi/4$

Problem 6: $\vec{v} = \langle -3\sin t, 3\cos t, 1\rangle, \quad \vec{a} = \langle -3\cos t, -3\sin t, 0\rangle, \quad \text{speed} = \sqrt{10}$

Problem 7: (a) $\vec{N} = \langle 4, -3, -2\rangle$

(b) $\vec{N} \cdot \vec{r}(t) = 6$.

(c)

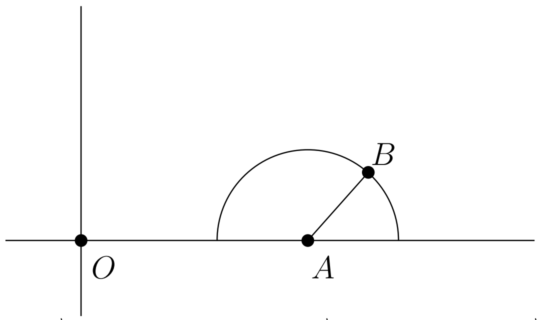
$$\frac{d}{dt}\Big(\vec{N} \cdot \vec{r}(t)\Big) = 0 \implies \frac{d\vec{N}}{dt} \cdot \vec{r}(t) + \vec{N} \cdot \frac{d\vec{r}}{dt} = 0 \implies \vec{N} \cdot \frac{d\vec{r}}{dt} = 0 \implies \vec{N} \cdot \vec{v}(t) = 0 \implies \vec{N} \perp \vec{v}(t).$$

Problem 8: $\vec{r}(t) = \langle t, \cos t, t\sin t\rangle$

Problem 9: $\vec{L}(u) = \langle 0, 3, 2u \rangle, \quad \vec{a}(0) = \langle 2, -3, 0 \rangle$

Problem 10: $t^3 + 2t$

Problem 11:



(a) $\overrightarrow{AB} = \langle \cos t, \sin t \rangle$ and $\overrightarrow{OA} = \langle 10t, 0 \rangle$, so $\overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{AB} = \langle 10t + \cos t, \sin t \rangle$.

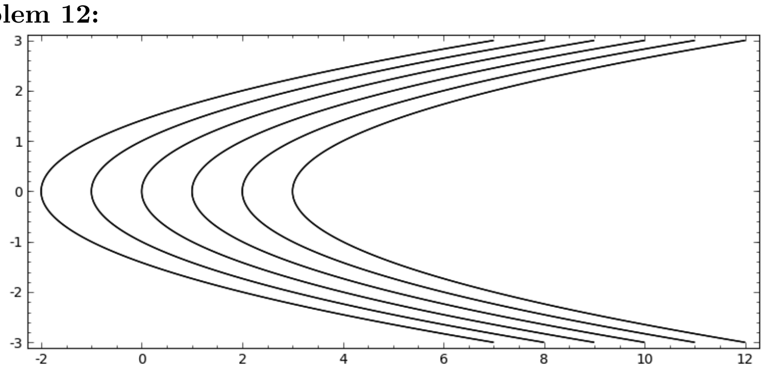
The rear bumper is reached at time $t = \pi$ and the position of B is $(10\pi - 1, 0)$.

(b) $\vec{v}(t) = \langle 10 - \sin t, \cos t \rangle$, so

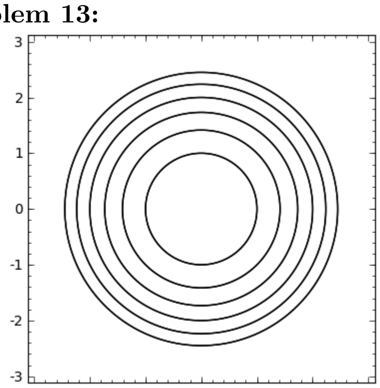
$$|\vec{v}|^2 = (10 - \sin t)^2 + \cos^2 t = 100 - 20 \sin t + \sin^2 t + \cos^2 t = 101 - 20 \sin t.$$

The speed is then given by $|\vec{v}| = \sqrt{101 - 20 \sin t}$.
The speed is smallest when $\sin t$ is largest i.e. $\sin t = 1$. It occurs when $t = \pi/2$. At this time, the position of the bug is $(5\pi, 1)$.
The speed is largest when $\sin t$ is smallest; that happens at the times $t = 0$ or π for which the position is then $(0, 0)$ and $(10\pi - 1, 0)$.

Problem 12:



Problem 13:



Problem 14: 4 (use change of variables $x^2 + y^2 + 1 = t$)

Problem 15: 1/4

Problem 16: (a) The horizontal trace for $z = 0$: the y-axis and the graph of the function $y = x^3$ in the xy-plane;
the vertical trace when $x = 0$: the y-axis in the yz-plane;
the vertical trace when $y = 0$: the graph of the function $z = -x^4$ in the xz-plane; this is a steeper upside-down parabola.

(b) $\langle -3, 1 \rangle$
(c) $\Delta w \approx -3\Delta x + \Delta y$

Problem 17: (a) $3x - y + 4z = 4$
(b) $(-1/4, 1/8, 1/8)$

Problem 18: (a) The volume is $xyz = xy(1 - x^2 - y^2) = xy - x^3y - xy^3$. Critical points: $f_x = y - 3x^2y - y^3 = 0, f_y = x - x^3 - 3xy^2 = 0$.
(b) $(x, y) = (1/2, 1/2)$
(c) local maximum
(d) 1/8

Problem 19: (a) $yz = 2\lambda x, xz = 2\lambda y, xy = \lambda$, and the constraint equation $x^2 + y^2 + z = 1$
(b) $x = 1/2, y = 1/2, z = 1/2$

Problem 20: $\dfrac{\partial w}{\partial x} = yf_u + \dfrac{1}{y}f_v$ and $\dfrac{\partial w}{\partial y} = xf_u - \dfrac{x}{y^2}f_v$.

Problem 21: (a) $\langle 3, 8 \rangle$
(b) $3x + 8y - z = 12$
(c) $f(1.9, 1.1) \approx 2.5$
(d) $\frac{5}{\sqrt{2}}$

Problem 22: (a) $(-20, 34)$ is a saddle point.

(b) There is no critical point in the first quadrant, hence the maximum must be at infinity or on the boundary of the first quadrant.
The boundary is composed of two half-lines:
- $x = 0$ and $y \geq 0$, on which $w = -y^2 - 12y$. It has a maximum ($w = 0$) at $y = 0$.
- $y = 0$ and $x \geq 0$, on which $w = -3x^2 + 16x$ (downwards parabola). It has a maximum when $w_x = -6x + 16 = 0 \implies x = 8/3$. Hence w has a local maximum at $(8/3, 0)$ and the value is $w = -3(8/3)^2 + 16(8/3) = 64/3 > 0$.

We now check that the maximum of w is not at infinity.
- $y \geq 0$ and $x \to \infty : w \leq -3x^2 + 16x$, which tends to $-\infty$ as $x \to +\infty$.
- $0 \leq x \leq C$ and $y \to \infty : w \leq -y^2 + 16C$, which tends to $-\infty$ as $y \to +\infty$.

We conclude that the maximum of w in the first quadrant is at $(8/3, 0)$.

Problem 23: (a) $w_x - \dfrac{y}{x^2}w_u + 2xw_v$ and $w_y = \dfrac{1}{x}w_u + 2yw_v$.
(b) $2vw_v$
(c) $10v^5$

Problem 24:

(a) $f(x,y,z) = x$ with constraint $g(x,y,z) = x^4+y^4+z^4+xy+yz+zx = 6$. The Lagrange multiplier equation is

$$\nabla f = \lambda \nabla g \Leftrightarrow \begin{cases} 1 = \lambda(4x^3 + y + z) \\ 0 = \lambda(4y^3 + x + z) \\ 0 = \lambda(4z^3 + x + y). \end{cases}$$

(b) The level surfaces of f and g are tangent at (x_0, y_0, z_0), so they have the same tangent plane. The level surface of f is the plane $x = x_0$; hence this is also the tangent plane to the surface $g = 6$ at (x_0, y_0, z_0).

Second method: at (x_0, y_0, z_0), we have $1 = \lambda g_x, 0 = \lambda g_y, 0 = \lambda g_z$. So $\lambda \neq 0$ and $\langle g_x, g_y, g_z \rangle = \langle 1/\lambda, 0, 0 \rangle$. This vector is therefore perpendicular to the tangent plane to the surface at (x_0, y_0, z_0). The equation of the plane is then $\frac{1}{\lambda}(x - x_0) = 0$, or equivalently $x = x_0$.

Problem 25: $\vec{L}(s) = \langle 1, 1, 1 \rangle + s\langle 16, -16, -4 \rangle$

Problem 26: $1/4$

Problem 29: $16/3$

Problem 30: $15\pi/8$

Problem 31: $11/24$

Problem 32:

$$\int_0^2 \int_0^{3\sqrt{1-\frac{x^2}{4}}} \int_0^{2+\sqrt{16-x^2-y^2}} y \, dz \, dy \, dx.$$

Problem 33: 6π

Problem 34:

$$\frac{8(\sqrt{3} - 1)}{3}$$

Problem 35:

$$\int_0^1 \int_{y/2}^{y} dx dy + \int_1^2 \int_{y/2}^{1} dx dy$$

Problem 36:
(a) $15/64$
(b) $(8/5, 7/15)$

Problem 37: 4π

Problem 38:
- cylindrical: $\dfrac{1}{\pi} \displaystyle\int_0^{2\pi} \int_0^1 \int_0^1 r\sqrt{r^2 + z^2} \, dz \, dr \, d\theta$
- rectangular: $\dfrac{1}{\pi} \displaystyle\int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_0^1 \sqrt{x^2 + y^2 + z^2} \, dz \, dy \, dx$

Problem 39: $\dfrac{16}{15\pi}$

Problem 40:

$$\int_0^{2\pi} \int_0^2 \int_{r^2}^4 3r^2 \cos\theta \; dz\, dr\, d\theta$$

Problem 41:

$$\int_0^{1/3} \int_0^{(1-3x)/2} \int_0^{1-3x-2y} z \; dz\, dy\, dx = \int_0^{1/3} \int_0^{(1-3x)/2} \frac{(1-3x-2y)^2}{2} \; dy\, dx$$

Problem 42: $192/5$

Problem 43: π

(Note that in order to make sense, the density has to be positive, so $0 < z < y$.)