MATH 20C – Midterm 1 Study Guide

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First, let me warn you that this is by no means a complete list of problems, or topics. Just highlights. The first thing you should do when preparing for the exam is to go through your notes, the online notes, the relevant sections of the book, the homework problems and the problems suggested in the homework section of the webpage `http://www.math.ucsd.edu/~alina/20c/hw.html`. If you still have trouble with some of the topics encountered so far, take the book (or another calculus book) and solve more problems related to that topic until you *really* understand how and why things work. The online homework system generates solutions after the due date for each problem set. Make full use of them!

Topics

vectors; operation with vectors (addition, subtraction, multiplication by scalars, dot product, cross product); determinants, area, volume; lines and planes in 3-space (normal vectors); decomposing a vector into a components along specified direction (projection, $\vec{v}_{||}$ and $\vec{v}_{\perp}$); parametric equations

calculus with vectors: limits, differentiation (product rules, chain rule), integration; velocity and acceleration vectors; speed and arc length; tangent line to a trajectory

Concept review

1. What are $\hat{\mathbf{\imath}}, \hat{\mathbf{\jmath}}, \hat{\mathbf{k}}$?
2. Write down the $\mathbf{v} = \langle 25, -3, 6 \rangle$ as a linear combination of $\hat{\mathbf{\imath}}, \hat{\mathbf{\jmath}}, \hat{\mathbf{k}}$. Explain to yourself in words what this vector means in terms of movement.
3. Formula for vector between two points.
4. Vector addition (formula and picture).
5. Vector subtraction (formula and picture).
6. Multiplication by scalar (formula and picture).
7. Write down a linear combination of your favorite 4 vectors in 3-space and write the result both in coordinates and in terms of $\hat{\mathbf{\imath}}, \hat{\mathbf{\jmath}}, \hat{\mathbf{k}}$.
8. What is the definition for the dot product of two vectors?
9. How do you find the cosine of the angle between two vectors?
10. What is the definition for the cross product? (formula and picture)
11. What does the cross product represent geometrically?
12. What is the area of a triangle in space? How about a parallelogram?
13. How do you find the sine of the angle between two vectors?

14. What is the determinant of three vectors?

15. What is the volume of the parallelepiped with sides $\mathbf{a}, \mathbf{b}, \mathbf{c}$?

16. What is the parametric equation for a line through two points?

17. What is the parametric equation for a line with direction $\mathbf{v}$ and passing through the point P_0?

18. What are the relative positions of two lines in space? Given two lines, how do you determine in which case you are?

19. Find the intersection of the line through $(1, 0, 10)$ with direction $\langle 2, -1, -1\rangle$ and the line through $(3, -4, 6)$ and $(-5, 2, 1)$.

20. What is the general equation of a plane?

21. What is the normal vector to the plane?

22. How can you find the normal vector the plane passing through three points P_1, P_2, P_3?

23. What is the equation of the plane passing through three points P_1, P_2, P_3?

24. What is the equation of the plane through P with normal vector $\mathbf{v}$?

25. What are the relative positions of a line and a plane? Given a line and a plane, how do you determine in which case you are?

26. Find the intersection of the line $\mathbf{L}(t) = \langle 1 + 2t, 4, 1 + t\rangle$ and the plane $2x + 3y - 4z = 10$.

27. How do you find the intersection of two planes?

28. What are the relative positions of two planes?

29. Find the intersection of $x + y + z = 5$ and $2x + 3y - 4z = 10$.

30. How do you find the parametric equation for the intersection of two surfaces?

31. Pick one such parametrization exercise from the textbook and do it.

Calculus with vectors

32. Write down the rule for taking $\lim_{t\to a} \mathbf{r}(t)$.

33. Write down the rule for differentiating $\mathbf{r}(t)$.

34. Write down the rules for integrating $\mathbf{r}(t)$.

35. What is the chain rule for vectors?

36. What is $\frac{d}{dt}(f(t)\mathbf{r}(t))$?

37. What is the rule for differentiating the dot product?

38. What is the rule for differentiating the cross product?

39. What is the velocity vector?

40. What is the acceleration? (vector or scalar? formula?)

41. What is the speed? (vector or scalar? formula?)

42. What is the arc length? (vector or scalar? formula?)

43. What is equation of the tangent line to a parametric curve $\mathbf{r}(t)$ at the point at time t_0?

Practice problems

This is a collection of extra problems to help you prepare for the midterm. The list is not complete and these problems do **not** imply anything about the content of the exam. A reasonable exam would consist of problems 2, 5, 8, 9 and 11. Try to solve these five problems in 50min. But also go through the rest. *Again, this does not mean the exam will have anything to do with problems 2, 5, 8, 9 or 11.*

1. Look over the homework so far (HW1–3) and the problems suggested on the homework webpage.

Section 12.1: 5, 19, 29, 37

Section 12.2: 1, 5, 27, 33, 53

Section 12.3: 5, 9, 13, 21, 29a, 33, 63

Section 12.4: 3, 9, 13, 17, 37,

Section 12.5: 1, 13, 17, 31, 49

Section 11.1: 1, 7, 11, 15, 19, 23, 25, 27, 53

Section 11.2: 1, 7, 9, 15, 21

Section 13.1: 3, 5, 9, 15, 19, 21, 27, 31, 33

Section 13.2: 1, 3, 7, 9, 13, 15, 17, 19, 21, 23, 29, 31, 33, 39, 45, 47, 51

Section 13.3: 1, 3, 5, 13, 23

Section 13.5: 3, 5, 7, 11, 19, 35

2. (a) Find the area of the space triangle with vertices $P_0 = (2, 1, 0), P_1 = (1, 0, 1), P_2 = (2, -1, 1)$.

(b) Find the equation of the plane containing the points P_0, P_1, P_2.

(c) Find the intersection of this plane with the line parallel to the vector $\vec{v} = \langle 1, 1, 1\rangle$ and passing through the point $S = (-1, 0, 0)$.

3. Let P, Q and R be the points at 1 on the x-axis, 2 on the y-axis and 3 on the z-axis, respectively.

(a) Express $\overrightarrow{QP}$ and $\overrightarrow{QR}$ in terms of $\hat{\mathbf{i}}, \hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$.

(b) Find the cosine of the angle between $\overrightarrow{QP}$ and $\overrightarrow{QR}$.

(c) Find the cosine of the angle at R of the triangle in space PQR.

4. Let $P = (1, 1, 1), Q = (0, 3, 1)$ and $R = (0, 1, 4)$.

(a) Find the area of the triangle in space PQR.

(b) Find the plane through P, Q and R expressed in the form $ax + by + cz = d$.

(c) Is the line through $(1, 2, 3)$ and $(2, 2, 0)$ parallel to this plane? Explain why or why not.

5. Find parametric equations for the tangent line to the curve $x = t^2 - 1, y = t^2 + 1, z = t + 1$ at the point $(-1, 1, 1)$.

6. At what angle do the lines $2x + y = 3$ and $3x - y = 4$ intersect?

7. The motion of a point P is given by the position vector $\vec{r}(t) = 3\cos t\hat{\mathbf{i}} + 3\sin t\hat{\mathbf{j}} + t\hat{\mathbf{k}}$. Compute the velocity, the acceleration and the speed of P.

8. (a) Find the normal vector $\vec{N}$ to the plane $4x - 3y - 2z = 6$.

(b) Let $P(t)$ be a point with position vector $\vec{r}(t)$. Express the property that $P(t)$ lies on the plane $4x - 3y - 2z = 6$ in vector notation as an equation involving $\vec{r}$ and the normal vector to the plane.

(c) By differentiating your answer to (b), show that $\vec{v}(t) = \dfrac{d\vec{r}}{dt}$ is perpendicular to the normal vector to the plane $\vec{N}$.

9. Find $\mathbf{r}(t)$ if $\mathbf{a}(t) = \langle 0, -\cos t, 2\cos t - t\sin t\rangle, \mathbf{v}(\pi) = \hat{\mathbf{i}} - \pi\hat{\mathbf{k}}, \mathbf{r}(\pi) = \pi\hat{\mathbf{i}} - \hat{\mathbf{j}}$.

10. Find the equation of the tangent line at $(0, 3, 0)$ of $\mathbf{r}(t) = \langle t\sin t, 3\cos t, t + \sin t\rangle$. What is the acceleration at the same point?

11. Find the arc length of the curve $\mathbf{r}(t) = \langle t^3, \sqrt{3}\,t^2, 2t + 8\rangle$ starting at $t_1 = 0$.

12. A ladybug is climbing on a Volkswagen Bug (= VW). In its starting position, the the surface of the VW is represented by the unit semicircle $x^2 + y^2 = 1$, $y \geq 0$ in the xy-plane. The road is represented as the x-axis. At time $t = 0$ the ladybug starts at the front bumper, $(1, 0)$, and walks counterclockwise around the VW at unit speed relative to the VW. At the same time the VW moves to the right at speed 10.

(a) Find the parametric formula for the trajectory of the ladybug, and find its position when it reaches the rear bumper. At $t = 0$, the rear bumper is at $(-1, 0)$.

(b) Compute the speed of the bug, and find where it is largest and smallest. *Hint: It is easier to work with the square of the speed.*