MATH 20C – Midterm 2 Study Guide

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First, let me warn you that this is by no means a complete list of problems, or topics. Just highlights. The first thing you should do when preparing for the exam is to go through your notes, the online notes, the relevant sections of the book and the homework problems. If you still have trouble with some of the topics encountered so far, take the book (or another calculus book) and solve more problems related to that topic until you *really* understand how and why things work.

Topics

functions of several variables: graphs, level curves and surfaces, contour maps; limits, partial derivatives, gradient, chain rule (I, II, III), directional derivatives

implicit differentiation, tangent planes to surfaces, tangent lines

optimization in several variables: critical points (local min/local max/saddle/degenerate), second derivative test, global min/max, Lagrange multipliers

Concept review

1. Make sure you know how to draw the level curves and the contour map of a function.
2. Limits of functions of several variables. How do you compute such a limit? How can you tell that a limit does not exist? Can you use the contour map for this?
3. What are the horizontal and vertical slices of the graph of $f(x, y)$?
4. What are the first order partial derivatives of a function of 2 variables? What about a function of 3 variables? What do they represent geometrically? How do you compute them?
5. Chain rule I, II and III (including the chain rule for paths using gradient and velocity).
6. What are the higher order partial derivatives of a function?
7. Compute $\frac{\partial^3 f}{\partial x \partial z^2}$ and $\frac{\partial^3 f}{\partial x \partial y \partial z}$ of $f(x, y, z) = xe^{-yz}$.
8. What is the gradient of a function?
9. Compute the gradient of $f(x, y, z) = xe^{-y^2 z}$.
10. What is the geometric interpretation of the gradient? (Hint: think normal vector.)
11. Write down the linear approximation formula for a function of 2 variables. And for a function of 3 variables.
12. Write down the equation of the tangent plane to the surface $f(x, y, z) = c$ at the point $P = (x_0, y_0, z_0)$.

13. Write down the equation of the tangent line to the curve $g(x, y) = c$ at the point $Q = (a, b)$.

14. How do you find the equation of the tangent line to a curve that is defined as the intersection of two surfaces? General method.

15. Now find the equation of the tangent line at $P = (2, 4, -1)$ to the curve obtained by intersecting $8x^2 - xy^2 + xz + yz - z^3 + 5 = 0$ and $x^3z + x^2yz^2 - xy^2 = -8$.

16. What are directional derivatives? How do you compute them? What is their geometric interpretation?

17. Implicit differentiation.

18. Compute $\frac{\partial x}{\partial z}$ at $(2, 4, -1)$ when $8x^2 - xy^2 + xz + yz - z^3 + 5 = 0$.

19. What is the Hessian matrix of $f(x, y)$?

20. Compute the Hessian of $f(x, y) = (x + y^2)\cos(x - y)$.

21. How do you find the critical points of a function?

22. Second derivative test.

23. Make sure you can determine the nature of a critical point (min/max/saddle) by looking at the contour map.

24. How do you find the global min/max of a function?

25. How do you find the min/max of $f(x, y, z)$ under the constraint $g(x, y, z) = c$? That is to say, make sure you understand Lagrange multipliers.

Practice problems

This is a collection of extra problems to help you prepare for the midterm. The list is not complete and these problems do **not** imply anything about the content of the exam. A reasonable exam would consist of problems 1–6 and 16. Try to solve these problems in 50min. But also go through the rest. *Again, this does not mean the exam will have anything to do with problems 1–6, 16.*

1. Look over the homework so far (HW5–7) and the problems suggested on the homework webpage.

   Section 14.1: 7, 21, 29, 31, 39

   Section 14.2: 1, 7, 9, 11

   Section 14.3: 9, 13, 21, 41, 43, 47, 61, 65

   Section 14.4: 3, 5, 7, 19, 21, 23, 29

   Section 14.5: 1, 5, 7, 11, 19, 21, 27, 29, 31, 33, 35, 37, 39, 41, 43, 45, 49, 60

   Section 14.6: 1, 3, 11, 25, 27

   Section 14.7: 1, 7, 9, 23, 29, 31, 35

   Section 14.8: 1, 5, 7, 9, 41

2. Draw the contour map for $f(x, y) = x - y^2$ showing at least six level curves.

3. Draw the contour map for $f(x, y) = \frac{1}{1+x^2+y^2}$ showing at least six level curves.

4. Evaluate $\displaystyle\lim_{(x,y)\to(0,0)} \frac{2(x^2 + y^2)}{\sqrt{x^2 + y^2 + 1} - 1}$.

5. Evaluate $\displaystyle\lim_{(x,y)\to(0,0)} \left(\frac{1}{2xy} - \frac{1}{xy(xy+2)} \right)$.

6. Let $f(x, y) = xy - x^4$.

   (a) Draw the horizontal and vertical traces of f through the origin.

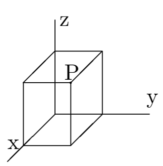
   (b) Find the gradient of f at $(1, 1)$.

   (c) Give an approximate formula telling how small changes Δx and Δy produce a small change Δw in the value of $w = f(x, y)$ at the point $(x, y) = (1, 1)$.

7. (a) Find the equation of the tangent plane to the surface $x^3y + z^2 = 3$ at the point $(-1, 1, 2)$.

   (b) Find the point(s) on the surface $3x^2-4y^2 = z$ at which the tangent plane is parallel to $3x+2y+2z = 7$.

8. A rectangular box is placed in the first octant as shown, with one corner at the origin and the three adjacent faces in the coordinate planes. The opposite point $P = (x, y, z)$ is constrained to lie on the paraboloid $x^2 + y^2 + z = 1$. The goal of this problem is to determine which P gives the box of greatest volume.



   (a) Show that the problem leads one to maximize $f(x, y) = xy - x^3y - xy^3$, and write down the equations for the critical points of f.

   (b) Find a critical point of f which lies in the first quadrant $(x > 0, y > 0)$.

   (c) Determine the nature of this critical point by using the second derivative test.

   (d) Find the maximum of f in the first quadrant (justify your answer).

9. In the problem above, instead of substituting for z, one could also use Lagrange multipliers to maximize the volume $V = xyz$ with the same constraint $x^2 + y^2 + z = 1$.

   (a) Write down the Lagrange multiplier equations for this problem.

   (b) Solve the equations (still assuming $x > 0, y > 0$).

10. Let $w = f(u, v)$, where $u = xy$ and $v = x/y$. Using the chain rule, express $\frac{\partial w}{\partial x}$ and $\frac{\partial w}{\partial y}$ in terms of x, y, f_u and f_v.

11. Let $f(x, y) = x^2y^2 - x$.

   (a) Find ∇f at $(2, 1)$.

   (b) Write the equation for the tangent plane to the graph of f at $(2, 1, 2)$.

   (c) Use a linear approximation to find the approximate value of $f(1.9, 1.1)$.

   (d) Find the directional derivative of f at $(2, 1)$ in the direction of $-\hat{\imath} + \hat{\jmath}$.

12. (a) Find the critical points of $w = -3x^2 - 4xy - y^2 - 12y + 16x$ and say what type each critical point is.

   (b) Find the point of the first quadrant $x \geq 0, y \geq 0$ at which w is largest. Justify your answer.

13. Let $u = y/x$, $v = x^2 + y^2$, $w = w(u, v)$.

    (a) Express the partial derivatives w_x and w_y in terms of x, y, w_u and w_v.

    (b) Express $xw_x + yw_y$ in terms of w_u and w_v. Write the coefficients as functions of u and v.

    (c) Find $xw_x + yw_y$ in case $w = v^5$.

14. (a) Find the Lagrange multiplier equations for the point of the surface $x^4+y^4+z^4+xy+yz+zx = 6$ at which x is largest. (Do not solve.)

    (b) Given that x is largest at the point (x_0, y_0, z_0), find the equation for the tangent plane to the surface at that point.

15. Suppose that $x^2 + y^3 - z^4 = 1$ and $z^3 + zx + xy = 3$. The two surfaces intersect in a curve $\mathcal{C}$. Find the equation of the tangent line to $\mathcal{C}$ at $(1, 1, 1)$.

16. Problems 1 and 24 from Section 14.7.

17. Problem 3 from *Preliminary questions,* Section 14.8.

18. Find the rate $\frac{\partial x}{\partial z}$ at which x changes when we change z and keep y constant at the point $(1, 0, 1)$ if the variables are subject to the constraint $x^3 + xz - z^2 + y^3 - \cos(xy) = 0$.