MATH 20C Lecture 2 - Tuesday, October 7, 2014

Observations

1. Two vectors pointing in the same direction are scalar multiples of each other.
2. The sum of three head-to-tail vectors in a triangle is 0.

Dot product

Definition $\vec{A} \cdot \vec{B} = a_1 b_1 + a_2 b_2 + a_3 b_3 + \ldots$ (a scalar, not a vector)

Geometrically $\vec{A} \cdot \vec{B} = |\vec{A}||\vec{B}| \cos\theta$, where θ is the angle between the two vectors.

Explained the result as follows. First, $\vec{A} \cdot \vec{A} = |\vec{A}|^2 \cos 0 = |\vec{A}|^2$ is consistent with the definition. Next, consider a triangle with sides $\vec{A}, \vec{B}$ and $\vec{C} = \vec{A} - \vec{B}$. Then the law of cosines gives $|\vec{C}|^2 = |\vec{A}|^2 + |\vec{B}|^2 - 2|\vec{A}||\vec{B}| \cos\theta$. On the other hand, we get

$$|\vec{C}|^2 = \vec{C} \cdot \vec{C} = (\vec{A} - \vec{B}) \cdot (\vec{A} - \vec{B}) = |\vec{A}|^2 + |\vec{B}|^2 - 2\vec{A} \cdot \vec{B}.$$

Hence the geometric interpretation is a vector formulation of the law of cosines.

Applications of the dot product

1. **Computing lengths and angles (especially angles)**: $\cos\theta = \frac{\vec{A} \cdot \vec{B}}{|\vec{A}||\vec{B}|}$.

   For instance, in the triangle in space with vertices at $P = (1,0,0), Q = (0,1,0), R = (0,0,2)$, the angle θ at P:

$$\cos\theta = \frac{\overrightarrow{PQ} \cdot \overrightarrow{PR}}{|\overrightarrow{PQ}||\overrightarrow{PR}|} = \frac{\langle -1,1,0 \rangle \cdot \langle -1,0,2 \rangle}{\sqrt{(-1)^2 + 1^2 + 0^2}\ \sqrt{(-1)^2 + 0^2 + 2^2}} = \frac{1}{\sqrt{10}}, \qquad \theta \approx 71.5^\circ.$$

2. **Relative direction of two vectors**

$$\operatorname{sign}(\vec{A} \cdot \vec{B}) = \begin{cases} > 0 & \text{, if } \theta < 90^\circ \text{(acute angle, the two vectors point more or less in} \\ & \text{the same direction)} \\ = 0 & \text{, if } \theta = 90^\circ \Leftrightarrow \vec{A} \perp \vec{B} \\ < 0 & \text{, if } \theta > 90^\circ \text{(obtuse angle, vectors point away from each other)} \end{cases}$$

3. **Detecting orthogonality:** it's worth emphasizing that $\vec{A} \perp \vec{B} = 0 \Leftrightarrow \vec{A} \cdot \vec{B} = 0$.

4. **Finding the component of a vector in a given direction** If $\vec{u}$ is a unit vector, the component of a vector A in the direction of $\vec{u}$ has length $|\vec{A}| \cos\theta = |\vec{A}||\vec{u}| \cos\theta = \vec{A} \cdot \vec{u}$ ($\theta =$ the angle between the two vectors). The component itself is a vector called the *projection* of $\vec{A}$ in the direction of $\vec{u}$, namely the vector

$$\vec{A}_{\parallel} = \mathbf{proj}_{\vec{u}}(\vec{A}) = (\vec{A} \cdot \vec{u})\vec{u}.$$

The component of $\vec{A}$ in the direction perpendicular to $\vec{u}$ is $\vec{A}_\perp = \vec{A} - \vec{A}_\|$, so

$$\vec{A} = \vec{A}_\| + \vec{A}_\perp.$$

Example: Find the component of $\vec{A} = \langle a_1, a_2, a_3 \rangle$ in the direction of the vector $\hat{\mathbf{i}} = \langle 1, 0, 0 \rangle$.

Step 1 Find the unit vector in the direction of $\hat{\mathbf{i}}$: we already have it, as $\hat{\mathbf{i}}$ has length 1.

Step 2 Find the length of the component:

$$|\vec{A}_\|| = \vec{A} \cdot \hat{\mathbf{i}} = \langle a_1, a_2, a_3 \rangle \cdot \langle 1, 0, 0 \rangle = a_1$$

Step 3 Multiply the results from step 1 and step 2:

$$\vec{A}_\| = a_1 \hat{\mathbf{i}} = \langle a_1, 0, 0 \rangle.$$

Application: pendulum.

5. **Planes**

   The plane $x + 2y + 3z = 0$ consists of the points $P = (x, y, z)$ with the property that $\vec{A} \cdot \overrightarrow{OP} = x + 2y + 3z = 0$, where $\vec{A} = \langle 1, 2, 3 \rangle$. This is the same as saying that $\vec{A} \perp \overrightarrow{OP}$.

Area

We can decompose the area of a polygon in the plane into a sum of areas of triangles. The area of the triangle with sides $\vec{A}$ and $\vec{B}$ is $\frac{1}{2}$base$\times$height $= \frac{1}{2}|\vec{A}||\vec{B}| \sin\theta = (\frac{1}{2}$area of the parallelogram).

So we need to compute $\sin\theta$. We know how to compute $\cos\theta$. Could do $\sin^2\theta + \cos^2\theta = 1$, but get ugly formula. Instead reduce to complementary angle $\theta' = \frac{\pi}{2} - \theta$ by considering $\vec{A}' = \vec{A}$ rotated by $90° = \frac{\pi}{2}$ counterclockwise (drew a picture).

Then, the area of the parallelogram with sides $\vec{A}, \vec{B}$ is $= |\vec{A}||\vec{B}| \sin\theta = |\vec{A}'||\vec{B}| \cos\theta' = \vec{A}' \cdot \vec{B}$
Continued from last time: If $\vec{A} = \langle a_1, a_2 \rangle$ and $\vec{A}' = \vec{A}$ rotated by $90° = \frac{\pi}{2}$ counterclockwise, what are the coordinates of $\vec{A}'$? (showed slide, multiple choice).

Answer: $\langle -a_2, a_1 \rangle$ (most students got it right).
So area of the parallelogram with sides $\vec{A}, \vec{B}$ is $= \vec{A}' \cdot \vec{B} = \langle -a_2, a_1 \rangle \cdot \langle b_1, b_2 \rangle = a_1 b_2 - a_2 b_1$.

Determinants in the plane

Definition: The determinant of vectors $\vec{A}, \vec{B}$ is $\det \begin{pmatrix} \vec{A} \\ \vec{B} \end{pmatrix} = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = a_1 b_2 - a_2 b_1$.

Geometrically: $\det \begin{pmatrix} \vec{A} \\ \vec{B} \end{pmatrix} = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = \pm$ area of the parallelogram. (Area is positive, determinant might be negative, so take absolute value.)

Determinants in space

Definition: The determinant of vectors $\vec{A}, \vec{B}, \vec{C}$ is

$$\det \begin{pmatrix} \vec{A} \\ \vec{B} \\ \vec{C} \end{pmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}.$$

Geometrically: $\det \begin{pmatrix} \vec{A} \\ \vec{B} \\ \vec{C} \end{pmatrix} = \pm$ the volume of the parellipiped with sides $\vec{A}, \vec{B}, \vec{C}$.

Cross-product

Is defined only for 2 vectors in space. Gives a vector (not a scalar, like dot product).

Definition: $\vec{A} \times \vec{B} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \hat{\mathbf{i}} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \hat{\mathbf{j}} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \hat{\mathbf{k}}$

(the 3×3 determinant is a *symbolic notation*, the actual formula is the expansion).

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Recall: $\vec{A} \times \vec{B} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \hat{\mathbf{i}} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \hat{\mathbf{j}} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \hat{\mathbf{k}}$

Geometrically: $\vec{A} \times \vec{B}$ is a vector with

- length: $|\vec{A} \times \vec{B}| =$ area of the parallelogram with sides $\vec{A}, \vec{B}$;
- direction: perpendicular on the plane containing $\vec{A}, \vec{B}$ and pointing in the direction given by the right hand rule.

Right hand rule:

1. extend right hand in direction of $\vec{A}$
2. curl fingers towards direction of $\vec{B}$
3. thumb points in same direction as $\vec{A} \times \vec{B}$

Question Compute $\hat{\mathbf{i}} \times \hat{\mathbf{j}} =$? (multiple choice) Answer: $\hat{\mathbf{k}}$ (most got it right). Checked both by picture and formula.

Properties of the cross product:

1. $\vec{B} \times \vec{A} = -\vec{A} \times \vec{B}$
2. $\vec{A} \times \vec{A} = 0$

Planes

1. The plane through 3 points, P_1, P_2, P_3.

   A point $P = (x, y, z)$ is in the plane if and only if the volume of the parallelipiped with sides $\overrightarrow{P_1P}, \overrightarrow{P_1P_2}, \overrightarrow{P_1P_3}$ has volume 0 (drew picture). This is the same as saying that

   $$\det \begin{pmatrix} \overrightarrow{P_1P} \\ \overrightarrow{P_1P_2} \\ \overrightarrow{P_1P_3} \end{pmatrix} = 0$$

   Another way:

   Note: In general the equation of a plane in space has the form

   $$\boxed{ax + by + cz = d}$$

2. The plane through the origin perpendicular to $\overrightarrow{N} = \langle 1, 5, 10 \rangle$

   Drew a picture. A point $P = (x, y, z)$ is in this plane if and only if $\overrightarrow{OP} \perp \overrightarrow{N}$, which is to say $\overrightarrow{OP} \cdot \overrightarrow{N} = 0$. This means

   $$\langle x, y, z \rangle \cdot \langle 1, 5, 10 \rangle = 0,$$

   which gives $x + 5y + 10z = 0$.

3. The plane through $P_0 = (2, 1, -1)$ and perpendicular to $\overrightarrow{N} = \langle 1, 5, 10 \rangle$

   Drew a picture. A point $P = (x, y, z)$ is in this plane if and only if $\overrightarrow{P_0P} \perp \overrightarrow{N}$, which is to say $\overrightarrow{P_0P} \cdot \overrightarrow{N} = 0$. This means

   $$\langle x - 2, y - 1, z + 1 \rangle \cdot \langle 1, 5, 10 \rangle = 0,$$

   which gives $x + 5y + 10z = -3$.

   This plane is parallel to the plane in the previous example. In both cases, the coefficients of x, y, z are the components of the vector $\overrightarrow{N}$.

   In the case of $x + 5y + 10z = -3$ one gets the constant -3 by plugging in the coordinates of the point P_0 in the left hand side.

4. Plane through $P_1 = (1, 0, 0), P_2 = (0, 1, 0), P_3 = (0, 0, 2)$. Normal vector:

   $$\vec{N} = \overrightarrow{P_1P_2} \times \overrightarrow{P_1P_3} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ -1 & 1 & 0 \\ -1 & 0 & 2 \end{vmatrix} = 2\hat{\mathbf{i}} - (-2)\hat{\mathbf{j}} + \hat{\mathbf{k}} = \langle 2, 2, 1 \rangle.$$

   Equation of the plane is $2x + 2y + z = 2$ (plug in P_1 for instance, but any of the P_i works).

Definition A vector perpendicular to a plane $\mathcal{P}$ is called a normal vector to that plane. Note that this is implies that all normal vectors to a given plane are proportional.

So the coefficients of x, y, z are the components of a normal vector to the plane. Conversely, if the equation of the plane is $ax + by + cz = d$, then $\langle a, b, c\rangle$ is a normal vector to it.

Note: Two planes are either parallel (if their normal vectors are proportional) or they intersect in a line.

Example: Are $\vec{v} = \langle 1, 2, -1\rangle$ and plane $x + y + 3z = 5$ parallel, perpendicular or neither?

Normal to plane: $\vec{N} = \langle 1, 1, 3\rangle$

A perpendicular vector would be proportional to $\langle 1, 1, 3\rangle$, i.e. to the coefficients; lets test if $\vec{v}$ is parallel to the plane: equivalent to being $\perp \vec{N}$. We have

$$\vec{v} \cdot \vec{N} = 1 + 2 - 3 = 0,$$

so $\vec{v}$ is parallel to the plane.

Parametric Equations

In general, parametric equations are a good way to describe arbitrary motions in plane or space. We have already seen an example of this earlier in the course, namely lines in space. It is convenient to think of trajectories in terms of the position vector $\vec{r}(t)$.

Lines

E.g. line through $Q_0 = (-1, 2, 2), Q_1 = (1, 3, -1)$: moving point $Q(t) = (x(t), y(t), z(t))$ starts at Q_0 at $t = 0$, moves at constant speed along line, reaches Q_1 at $t = 1$: so $\overrightarrow{Q_0Q(t)} = t\overrightarrow{Q_0Q_1}$.

Plug in our Q_0 and Q_1 : $\langle x(t) + 1, y(t) + 2, z(t) - 2\rangle = t\langle 2, 1, -3\rangle$, i.e.

$$\begin{cases} x(t) &= -1 + 2t \\ y(t) &= -2 + t \\ z(t) &= 2 - 3t \end{cases}$$

In terms of position vector $\vec{r}(t) = \vec{O}Q(t) = \langle x(t), y(t), z(t)\rangle$, the parametric equation reads

$$\vec{r}(t) = \langle -1, -2, 2\rangle + t\langle 2, 1, -3\rangle = \overrightarrow{OQ_0} + t\,\overrightarrow{Q_0Q_1}.$$

Lines and planes

Understand where lines and planes intersect.

Intersection of line through Q_0, Q_1 with plane $x + 2y + 4z = 7$? When does the moving point $Q(t)$ lie in the plane? Check: at $Q(t)$,

$$x + 2y + 4z = (-1 + 2t) + 2(2 + t) + 4(2 - 3t) = 11 - 8t,$$

so condition is $11 - 8t = 7$ or $t = 1/2$. Intersection point: $Q(t = 1/2) = (0, 5/2, 1/2)$.

Relative positions of a line and a plane:

To figure it out, take the parametric equation of the line and plug into the equation of the plane. Again, 3 possibilities:

1. the line is parallel the plane (no solutions)
2. the line is contained in the plane (infinitely many solutions)
3. the line intersects the plane in a point (one solution)

General parametric equations

Example: $\vec{r}(t) = \langle \cos t, \sin t \rangle$ describes a circle in the plane of radius 1, centered at the origin.

Question: What if we take $\vec{r}(t) = \langle \cos(2t), \sin(2t) \rangle$? Do we still get the unit circle?

The answer is "yes", but the point moves on the trajectory twice as fast.