MATH 20C Lecture 3 - Tuesday, October 14, 2014

More parametric equations

1. $\vec{r}(t) = \langle 1+t, 2+t \rangle$ describes a line in the plane in the direction of the vector $\langle 1, 1 \rangle$, through the point $(1, 2)$.

2. $\vec{r}(t) = \langle 1+t^2, 2+t^2 \rangle$

   *Question*Does this describe a 1) line? 2)half-line? 3) circle? 4) parabola (most got it right)

   Answer: half- line

   To see this, the components are

   $$x = 1 + t^2$$

   $$y = 2 + t^2$$

   Elliminate the parameter t and get $y = x + 1$ but only points with coordinates at least $(1, 2)$.

   Beware! The parametric equation is not unique. That is to say, the same curve in plane or space can be described by many different parametric equations.

Now the other way around, let's find the parametric equation for a given trajectory.
Question: How to find the parametric equation for a given trajectory?
Example: Find the circle in the plane of radius 5 centered $P = (1, 3)$.
First, the circle of radius 5 centered at the origin has the parametric equation $\langle 5\cos t, 5\sin t \rangle$. In ordered to obtain the desired circle, just translate by $\overrightarrow{OP}$. So we get

$$\vec{r}(t) = \langle 1 + 5\cos t, 3 + 5\sin t \rangle.$$

Another example: Find a parametric equation for the circle of radius 5 centered at $P = (1, 6, 8)$ lying in a plane parallel to the xz-plane.
We'll follow the same path as before.

Step 1: Write down a parametric equation for the circle centered at the origin, of radius 5, in the xz-plane. Get $\langle 5\cos t, 0, 5\sin t \rangle$.

Step 2: Translate by the vector $\overrightarrow{OP}$. We get the answer

$$\vec{r}(t) = \langle 1 + 5\cos t, 6, 8 + 5\sin t \rangle.$$

Intersection of surfaces

Another important way to get curves, is by taking the intersection of two surfaces.

We already know an example, namely the intersection of two planes. Going back to the example from the beginning of the lecture, let's find a parametric equation for the intersection of the first 2 planes,

$$\begin{aligned} x+y+2z &= 7 \\ x+y-4z &= 9 \end{aligned}$$

We parametrize in terms of $t=z$. That gives

$$\begin{aligned} x+y &= 7-2t \\ x-y &= 9+4t \end{aligned}$$

Solve and get $x=8+t, y=-1-3t$, therefore $\vec{r}(t)=\langle 8,-1,0\rangle+t\langle 1,-3,1.\rangle$.

Another example: Parametrize the intersection of the surfaces

$$\begin{aligned} &x^2-y^2=z-1 \\ &x^2+y^2=4. \end{aligned}$$

We'll do it in two ways.

First way Choose parameter $x=t$. That gives $y^2=4-t^2$ and $z=-3+2t^2$. The problem is that when we solve for y we get two different solutions $y=\pm\sqrt{4-t^2}$. So we need two parametrizations to describe the whole curve:

$$\vec{r}_1(t)=\langle t,\sqrt{4-t^2},2t^2-3\rangle$$

and

$$\vec{r}_2(t)=\langle t,-\sqrt{4-t^2},2t^2-3\rangle.$$

Second way Parametrize the second curve by $x=2\cos t, y=2\sin t$ and plug into the first equation. Get $z=1+4\cos^2 t-4\sin^2 t=1+4\cos(2t)$, so

$$\vec{r}(t)=\langle 2\cos t,2\sin t,1+4\cos(2t)\rangle.$$

The position vector of a particle moving along a trajectory in plane [or space] is $\vec{r}(t)=x(t)\mathbf{\hat{\imath}}+y(t)\mathbf{\hat{\jmath}}$ $[+z(t)\mathbf{\hat{k}}]$.

The velocity vector is $\vec{v}(t)=\dfrac{d\vec{r}}{dt}=\vec{r}\,'(t)=\left\langle\dfrac{dx}{dt},\dfrac{dy}{dt},\dfrac{dz}{dt}\right\rangle$ in space or $\vec{v}(t)=\left\langle\dfrac{dx}{dt},\dfrac{dy}{dt}\right\rangle$in the plane.

Note: We take derivatives, limits and integrate vectors **componentwise**.

The cycloid: curve traced by a point on a wheel of radius 1 that is rolling on a flat surface at unit speed. It has equation

$$\vec{r}(t)=\langle t-\sin t,1-\cos t\rangle.$$

The velocity vector is $\vec{v}(t) = \langle 1 - \cos t, \sin t \rangle$. At $t = 0$, $\vec{v} = \vec{0}$: translation and rotation motions cancel out, while at $t = \pi$ they add up and $\vec{v} = \langle 2, 0 \rangle$.

The *speed* is defined as the magnitude of the velocity vector. In this case,

$|\vec{v}| = \sqrt{(1 - \cos t)^2 + \sin^2 t} = \sqrt{2 - 2\cos t}$ (smallest at $t = 0, 2\pi, \ldots$, largest at $t = \pi$).

Remark: The speed is $\left|\frac{d\vec{r}}{dt}\right|$ which is NOT the same as $\frac{d|\vec{r}|}{dt}$!

Acceleration: $\vec{a}(t) = \dfrac{d\vec{v}}{dt} = \vec{r}\,''(t)$. E.g., cycloid: $\vec{a}(t) = \langle \sin t, \cos t \rangle$ (at $t = 0$ $\vec{a} = \langle 0, 1 \rangle$ is vertical).

Example: The circle $\vec{r}(t) = \langle \cos t, \sin t \rangle$.

The velocity vector is $\vec{v}(t) = \langle -\sin t, \cos t \rangle$, and the speed is $|\vec{v}| = 1$ for any t. The acceleration vector is $\vec{a} = \langle -\cos t, -\sin t \rangle$.

Arc length

$s =$ distance travelled along trajectory. Since the rate of change of the distance is the speed, we have $\frac{ds}{dt} =$ speed $= |\vec{v}(t)|$. Can recover length of trajectory by integrating ds/dt. So the length of the curve starting at time t_1 until time t_2 is

$$s = \int_{t_1}^{t_2} |\vec{v}(t)| dt.$$

The distance traveled by a moving point along a curve starting at time t_1 until the current time is

$$s(t) = \int_{t_1}^{t} |\vec{v}(u)| du.$$

Computing such an integral is not always easy... Sometimes it can be done, though. Here are a few examples.

1. One rotation of the wheel for the cycloid: $s = \int_0^{2\pi} \sqrt{2 - 2\cos t}\ dt$.
2. $\vec{r}(t) = \langle \cos t, \sin t \rangle$

   Then the velocity is $\vec{v} = \langle -\sin t, \cos t \rangle$ and the speed is $|\vec{v}(t)| = 1$ (contstant). Arc length starting at $t_1 = 0$ and ending at $t_2 = 2\pi$ is

   $$s = \int_0^{2\pi} 1 dt = 2\pi$$

Integration

$$\int \vec{r}(t) dt = \left\langle \int x(t) dt, \int y(t) dt, \int z(t) dt \right\rangle + \vec{c}, \qquad \vec{c} = \langle c_1, c_2, c_3 \rangle$$

Example: $\vec{a}(t) = \hat{\mathbf{k}}, \vec{v}(0) = \hat{\mathbf{\imath}}, \vec{r}(0) = \hat{\mathbf{\jmath}}$. Find $\vec{r}(t)$.

Since $\vec{a}(t) = \frac{d\vec{v}}{dt}$, that means that $\vec{v}(t) = t\hat{\mathbf{k}} + \vec{c}$. And now $\vec{v}(0) = \hat{\mathbf{i}}$, tells us $\vec{c} = \hat{\mathbf{i}}$. Therefore

$$\vec{v}(t) = t\hat{\mathbf{k}} + \hat{\mathbf{i}}.$$

Repeating the procedure for $\vec{v}(t) = \frac{d\vec{r}}{dt}$, we get

$$\vec{r}(t) = \frac{t^2}{2}\hat{\mathbf{k}} + t\hat{\mathbf{i}} + \hat{\mathbf{j}} = \langle t, 1, t^2/2 \rangle.$$

MATH 20C Lecture 5 - Thursday, October 16, 2014

Unit tangent vector

$$\hat{\mathbf{T}} = \frac{\vec{v}}{|\vec{v}|}$$

We have $\dfrac{d\vec{r}}{dt} = \dfrac{d\vec{r}}{ds}\dfrac{ds}{dt} = \hat{\mathbf{T}}|\vec{v}|$ (here s = arc length)

Tangent lines

The tangent line at time t_0 to the trajectory $\vec{r}(t)$ is the line through the point $P = (x(t_0), y(t_0), z(t_0))$ in the direction given by the velocity vector at that point, $\vec{v}(t_0)$. It has parametric equation

$$\vec{L}(s) = \vec{r}(t_0) + s\vec{v}(t_0).$$

Note: The parameter for the tangent line is different from the parameter of the curve itself.

Take for instance $\vec{r}(t) = \langle t, 1, t^2 \rangle$. Then the tangent line at time $t = 1$ is the line that passes through the point $P = (1, 1, 1)$ and has direction given by the velocity vector at $t = 1$, $\vec{v}(1) = \langle 1, 0, 2 \rangle$.

Hence

$$\vec{L}(\theta) = \langle 1, 1, 1 \rangle + \theta \langle 1, 0, 2 \rangle.$$

Note that at $\theta = 0$ the line touches the curve, and then it goes away from it.

Example 1 Set up an integral, but do not evaluate, for the arc length starting at time 5 of the curve $\vec{r}(t) = \langle \frac{2}{3}t^3, 4t, -\sqrt{2}t^2 \rangle$.

We have $\vec{v}(t) = \langle 2t^2, 4, -2\sqrt{2}t \rangle$, and

$$s(t) = \int_5^t \sqrt{4t^4 + 16 + 8t^2}\, dt.$$

Example 2 $\vec{r}(t) = 2t\hat{\mathbf{i}} + (\ln t)\hat{\mathbf{j}} + t^2\hat{\mathbf{k}}$

Then $\vec{v}(t) = 2\hat{\mathbf{i}} + \frac{1}{t}\hat{\mathbf{j}} + 2t\hat{\mathbf{k}}$, and the arc length starting at $t_1 = 1$ is

$$s(t) = \int_1^t \sqrt{4 + \frac{1}{u^2} + 4u^2}\, du = \int_1^t \sqrt{\left(2u + \frac{1}{u}\right)^2} du = \int_1^t \left(2u + \frac{1}{u}\right) du = t^2 - 1 + \ln t.$$

Example 3 $\vec{a}(t) = \hat{\mathbf{k}}, \vec{v}(0) = \hat{\mathbf{i}}, \vec{r}(0) = \hat{\mathbf{j}}$. Find $\vec{r}(t)$.

Since $\vec{a}(t) = \frac{d\vec{v}}{dt}$, that means that $\vec{v}(t) = \int \vec{a}(t)dt = t\hat{\mathbf{k}} + \vec{c}$. (Note that we integrate componentwise). And now $\vec{v}(0) = \hat{\mathbf{i}}$, tells us $\vec{c} = \hat{\mathbf{i}}$. Therefore

$$\vec{v}(t) = t\hat{\mathbf{k}} + \hat{\mathbf{i}}.$$

Repeating the procedure for $\vec{v}(t) = \frac{d\vec{r}}{dt}$, we get

$$\vec{r}(t) = \frac{t^2}{2}\hat{\mathbf{k}} + t\hat{\mathbf{i}} + \hat{\mathbf{j}} = \langle t, 1, t^2/2 \rangle.$$

Example 4 $\vec{a}(t) = 2\hat{\mathbf{i}} + 2t\hat{\mathbf{j}}, \vec{v}(1) = 9\hat{\mathbf{i}} + \hat{\mathbf{j}}, \vec{r}(1) = 10\hat{\mathbf{i}} + \frac{1}{3}\hat{\mathbf{j}} + 9\hat{\mathbf{k}}$. Find $\vec{r}(t)$.

Again, $\vec{v}(t) = \int \vec{a}(t)dt = 2t\hat{\mathbf{i}} + t^2\hat{\mathbf{j}} + \vec{c}$. Using the initial condition $\vec{v}(1) = \langle 9, 1, 0 \rangle$ gives

$$\vec{v}(t) = \langle 2t + 7, t^2, 0 \rangle.$$

Integrate and obtain $\vec{r}(t) = \int \vec{v}(t)dt = \left\langle t^2 + 7t, \frac{t^3}{3}, 0 \right\rangle + \vec{d}$. The initial condition implies $\vec{d} = \langle 2, 0, 9 \rangle$ so

$$\vec{r}(t) = \left\langle t^2 + 7t + 2, \frac{t^3}{3}, 9 \right\rangle = (t^2 + 7t + 2)\hat{\mathbf{i}} + \frac{t^3}{3}\hat{\mathbf{j}} + 9\hat{\mathbf{k}}.$$

Example 5 example 3, section 13.5