# MATH 20C Lecture 7 - Tuesday, October 28, 2014

## Functions of several variables

Recall: for a function of 1 variable, we can plot its graph, and the derivative is the slope of the tangent line to the graph. Plotting graphs of functions of 2 variables: examples $z = -y$, $z = 1 - x^2 - y^2$, using slices by the coordinate planes. (derived carefully). Contour map: level curves $f(x, y) = c$. Amounts to slicing the graph by horizontal planes $z = c$.

Showed 2 examples from "real life": a topographical map, and a temperature map, then did the examples $z = -y$ and $z = 1 - x^2 - y^2$.
Contour map gives some qualitative info about how $f$ varies when we change $x, y$. (shown an example where increasing $x$ leads $f$ to increase, but increasing $y$ leads $f$ to decrease).

## Limits

By substitution:

$$\lim_{(x,y)\to(0,0)} \frac{3x^2 + 2y - 5\cos(4(x+y))}{xy - e^{x-y}} = \frac{0 + 0 - 5}{0 - 1} = 5$$

Disclaimer: limits do not always exist!
For instance, take $\lim_{(x,y)\to(0,0)} \frac{x^2}{x^2+y^2}$. Direct substitution does not work. Drawn contour map. We see that a bunch of level curves intersect at $(0, 0)$, so the limit does not exist.

On the other hand $\lim_{(x,y)\to(0,0)} \frac{\sin(x+y)}{(x+y)} = 1$.

# MATH 20C Lecture 8 - Thursday, October 30, 2014

## Partial derivatives

$$f_x = \frac{\partial f}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x, y_0) - f(x_0, y_0)}{\Delta x}; \text{ same for } f_y.$$

Geometric interpretation: $f_x, f_y$ are slopes of tangent lines of vertical slices of the graph of $f$ (fixing $y = y_0$; fixing $x = x_0$).
How to compute: treat $x$ as variable, $y$ as constant.
Example: $f(x, y) = x^3y + y^2$, then $f_x = 3x^2y, f_y = x^3 + 2y$.

Another example: $g(x, y) = \cos(x^3y + y^2)$.

Use chain rule (version I)

$$\boxed{\frac{\partial F}{\partial x} = \frac{dF}{du}\frac{\partial u}{\partial x}}$$

Here $F(u) = \cos u$ and $u = f$, so get $\frac{\partial g}{\partial x} = -(3x^2y)\sin(x^3y + y^2)$.

Product rule:

$$\boxed{\frac{\partial (fg)}{\partial x}(x_0, y_0) = g(x_0, y_0)\frac{\partial f}{\partial x} + f(x_0, y_0)\frac{\partial g}{\partial x}}$$

**Linear approximation**

Linear approximation formula:

$$\Delta f \approx f_x \Delta x + f_y \Delta y.$$

Justification: $f_x$ and $f_y$ give slopes of two lines tangent to the graph:

$$L_1 : \begin{cases} y = y_0 \\ z = z_0 + f_x(x_0, y_0)(x - x_0) \end{cases} \quad \text{and } L_2 : \begin{cases} x = x_0 \\ z = z_0 + f_y(x_0, y_0)(y - y_0). \end{cases}$$

We can use this to get the equation of the tangent plane to the graph:

$$z = z_0 + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0).$$

Approximation formula says that the graph is close to its tangent plane.

Recall chain rule I: $g = F(u)$ and $u = u(x, y)$, then $\frac{\partial g}{\partial x} = \frac{dF}{du}\frac{\partial u}{\partial x}$. Used this to compute the partial derivatives of $g(x, y, z) = \ln(x^2 + y^2 - xz)$. Get

$$\frac{\partial g}{\partial x} = \frac{2x - z}{x^2 + y^2 - xz}, \quad \frac{\partial g}{\partial z} = \frac{-x}{x^2 + y^2 - xz}.$$

## Higher order partial derivatives

Are computed by taking successive partial derivatives. For instance $\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)$ and so on.

Computed

$$\begin{aligned} \frac{\partial^2 g}{\partial z \partial x} &= \frac{\partial x}{\partial z}\left(\frac{\partial g}{\partial x}\right) = \frac{\partial}{\partial z}\left(\frac{2x - z}{x^2 + y^2 - xz}\right) = \frac{(-1)(x^2 + y^2 - xz) - (2x - z)(-x)}{(x^2 + y^2 - xz)^2} \\ \frac{\partial^2 g}{\partial x \partial z} &= \frac{\partial}{\partial x}\left(\frac{\partial g}{\partial z}\right) = \frac{\partial}{\partial x}\left(\frac{-x}{x^2 + y^2 - xz}\right) = \frac{(-1)(x^2 + y^2 - xz) - (-x)(2x - z)}{(x^2 + y^2 - xz)^2} \end{aligned}$$

Notice that $\frac{\partial^2 g}{\partial z \partial x} = \frac{\partial^2 g}{\partial x \partial z}$. This is no coincidence. In general,

$$\boxed{\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}}$$

## Chain rule with more variables

Chain rule II: For example $w = f(x, y)$, $x = x(u, v), y = y(u, v)$. Then we can view $f$ as a function of $u$ and $v$.

$$\boxed{\begin{aligned} \frac{\partial f}{\partial u} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u} \\ \frac{\partial f}{\partial v} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v} \end{aligned}}$$

The idea behind each formula is that changing $u$ causes both $x$ and $y$ to change, at rates $\partial x/\partial u$ and $\partial y/\partial u$. The change in $x$ affects $f$ at the rate of $\partial f/\partial x$, for a total effect of $\frac{\partial f}{\partial x}\frac{\partial x}{\partial u}$. At the same time, the change in $y$ affects $f$ at the rate of $\partial f/\partial y$, for a total effect of $\frac{\partial f}{\partial y}\frac{\partial y}{\partial u}$. Finally, the two effects add up to produce the change in $f$ given by the first line in the boxed formula.

## Gradient vector

Chain rule III: function $f = f(x, y, z)$ and $x = x(t), y = y(t), z = z(t)$ :

$$\frac{df}{dt} = f_x \frac{dx}{dt} + f_y \frac{dy}{dt} + f_z \frac{dz}{dt} = \nabla f \cdot \left\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right\rangle$$

where $\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle$ is called the *gradient vector* of $f(x, y, z)$. Using this notation, the chain rule can be re-written as follows. On the path described by $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$, we have

$$\frac{df}{dt} = f_x \frac{dx}{dt} + f_y \frac{dy}{dt} + f_z \frac{dz}{dt} = \nabla f \cdot \left\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right\rangle.$$

That is,

$$\boxed{\frac{df}{dt} = \nabla f \cdot \frac{d\vec{r}}{dt} = \nabla f \cdot \vec{v}}$$

where $\vec{v}$ is the velocity vector.
Note: $\nabla f$ is a vector whose value depends on the point $(x, y, z)$ where we evaluate $f$.