MATH 20C Lecture 9 - Tuesday, November 4, 2014

Gradient vector

Recall: the gradient vector of $f(x, y, z)$ is $\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle$.

Theorem: ∇f is perpendicular to the level surfaces $f = c$.

Proof: take a curve $\vec{r} = \vec{r}(t)$ contained inside level surface $f = c$. Then velocity $\vec{v} = d\vec{r}/dt$ is in the tangent plane, and by chain rule, $dw/dt = \nabla f \cdot \vec{v} = 0$, so $\vec{v} \perp \nabla f$. This is true for every $\vec{v}$ in the tangent plane.

Example 1: $f(x, y, z) = a_1x + a_2y + a_3z$, then $\nabla f = \langle a_1, a_2, a_3 \rangle$. The level surface $f = c$ is $a_1x + a_2y + a_3z = c$. This is a plane with normal vector $\langle a_1, a_2, a_3 \rangle = \nabla f$, so ∇f is perpendicular on the plane $f(x, y, z) = c$.

Example 2: $f(x, y) = x^2 + y^2$, then $f = c$ are circles, $\nabla w = \langle 2x, 2y \rangle$ points radially out so $\perp$ circles.

Application: the tangent plane to a surface $f(x, y, z) = c$ at a point P is the plane through P with normal vector $\nabla f(P)$.

Example: tangent plane to $x^2 + y^2 - z^2 = 4$ at $(2, 1, 1)$: gradient is $\langle 2x, 2y, -2z \rangle = \langle 4, 2, -2 \rangle$; tangent plane is $4x + 2y - 2z = 8$. (Here we could also solve for $z = \pm\sqrt{x^2 + y^2 - 4}$ and use linear approximation formula, but in general we can't.)

Another way to get the tangent plane: $\Delta f \approx 4\Delta x + 2\Delta y - 2\Delta z$. On the level surface we have $\Delta f = 0$, so its tangent plane approximation is $4\Delta x + 2\Delta y - 2\Delta z = 0$, i.e. $4(x-2)+2(y-1)-2(z-1) = 0$, same as above.

Directional derivatives

We want to know the rate of change of f as we move (x, y) in an arbitrary direction.

Take a unit vector $\hat{u}$ and look at the cross-section of the graph of f by the vertical plane parallel to $\hat{u}$ and passing through the point (x, y). This is a curve passing through the point $P = (x, y, z = f(x, y))$ and we want to compute the slope the tangent line to this curve at P.

Notice that $\frac{\partial f}{\partial x}$ is the directional derivative in the direction of $\hat{\imath}$ and $\frac{\partial f}{\partial y}$ is the directional derivative in the direction of $\hat{\jmath}$.

Notation: $D_{\hat{u}} f(x_0, y_0)$ denotes the derivative of f in the direction of the unit vector $\hat{u}$ at the point (x_0, y_0).

Shown $f = x^2 + y^2 + 1$, and rotating slices through a point of the graph.

How to compute

Say that $\hat{u} = \langle a, b\rangle$. In order to compute $D_{\hat{u}} f(x_0, y_0)$, look at the straight line trajectory $\vec{r}(s)$ through (x_0, y_0) with velocity $\hat{u}$ given by $x(s) = x_0 + as, y(s) = y_0 + bs$. Then by definition $D_{\hat{u}} f(x_0, y_0) = \dfrac{df}{ds}$. This we can compute by chain rule to be $\dfrac{df}{ds} = \nabla f \cdot \dfrac{d\vec{r}}{ds}$. Hence

$$\boxed{D_{\hat{u}} f(x_0, y_0) = \nabla f(x_0, y_0) \cdot \hat{u}.}$$

Example Compute the directional derivative of $f = x^2 + y^2 - z^2$ at $P = (2, 1, 1)$ in the direction of $\hat{u} = \langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\rangle$.

so $\nabla f(P) = \langle 4, 2, -2\rangle$.
The unit vector in the direction of $\hat{u}$ is $\hat{u}$ itself. So $D_{\hat{u}} f(P) = \nabla f(P) \cdot \hat{u} = 3\sqrt{2}$. Therefore f is increasing in the direction of $\hat{u}$.

Geometric interpretation: $D_{\hat{u}} f = \nabla f \cdot \hat{u} = |\nabla f| \cos\theta$. Maximal for $\cos\theta = 1$, when $\hat{u}$ is in direction of ∇f. Hence: direction of ∇f is that of fastest increase of f, and $|\nabla f|$ is the directional derivative in that direction.

It is minimal in the opposite direction.

We have $D_{\hat{u}} f = 0$ when $\hat{u} \perp \nabla f$, i.e. when $\hat{u}$ is tangent to direction of level surface.

Implicit differentiation

Example: $x^2 + yz + z^3 = 8$. Viewing $z = z(x, y)$, compute $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$.

Take $\frac{\partial}{\partial x}$ of both sides of $x^2 + yz + z^3 = 8$. Get $2x + y\frac{\partial z}{\partial x} + 3z^2\frac{\partial z}{\partial x} = 0$, hence $\frac{\partial z}{\partial x} = -\frac{2x}{y+3z^2} = -\frac{2}{3}$.

In general, consider a surface $F(x, y, z) = c$. The we can view $z = z(x, y)$ as a function of two independent variables x, y and compute $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$. To do so, we take the partial derivative with respect to x of both sides of the equation $F(x, y, z) = c$ and get (by the chain rule)

$$\frac{\partial F}{\partial x}\frac{\partial x}{\partial x} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial x} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x} = 0.$$

But $\partial x/\partial x = 1$ and, since x and y are independent, $\partial y/\partial x = 0$ (changing x does not affect y). Hence the equation above really says that $F_x + F_z\frac{\partial z}{\partial x} = 0$ which implies

$$\boxed{\frac{\partial z}{\partial x} = -\frac{F_x}{F_z}.}$$

Similarly,

$$\boxed{\frac{\partial z}{\partial y} = -\frac{F_y}{F_z}.}$$

Changing gears, let's see how we can recover f from its gradient. Say $\nabla f = \langle 3x^2y, x^3+2z, 2y+\cos z\rangle$.

We proceed by successive integration.

We are given that $f_x = 3x^2y$. Integrating with respect to x (view y, z as constants), we see that $f = x^3y + g(y,z)$. Therefore

$$f_y = x^3 + \frac{\partial g}{\partial y}.$$

But we know from the gradient that $f_y = x^3 + 2z$, hence $g_y = 2z$.

Integrate with respect to y and get $g = 2yz + h(z)$, hence $f = x^3y + 2yz + h(z)$.

Since $f_z = 2y + \cos z$ we get that $\frac{dh}{dz} = \cos z$, so $h(z) = \sin z + C$. Substituting in the expression of f gives $f = x^3y + 2yz + \sin z + C$.

MATH 20C Lecture 10 - Thursday, November 6, 2014

Min/max in several variables

At a local max or min, $f_x = 0$ and $f_y = 0$ (since (x_0, y_0) is a local max or min of the slice). Because 2 lines determine tangent plane, this is enough to ensure that the tangent plane is horizontal.

Definition A critical point of f is a point (x_0, y_0) where $f_x = 0$ and $f_y = 0$. A critical point may be a local min, local max, or saddle. Or degenerate. Pictures shown of each type. To decide, apply **second derivative test**.

Example: $f(x,y) = x^2 - 2xy + 3y^2 + 2x - 2y$.
Critical point: $f_x = 2x - 2y + 2 = 0, f_y = -2x + 6y - 2 = 0$, gives $(x_0, y_0) = (-1, 0)$ (only one critical point).
Definition The hessian matrix of f is

$$H(x,y) = \begin{bmatrix} \dfrac{\partial^2 f}{\partial x^2} & \dfrac{\partial^2 f}{\partial y \partial x} \\ \\ \dfrac{\partial^2 f}{\partial x \partial y} & \dfrac{\partial^2 f}{\partial y^2} \end{bmatrix}.$$

Second derivative test

Let (x_0, y_0) be a critical point of f.

Case 1 $\det H > 0, f_{xx} > 0$: (x_0, y_0) is a local minimum

Case 2 $\det H > 0, f_{xx} < 0$: (x_0, y_0) is a local maximum

Case 3 $\det H < 0$: (x_0, y_0) is a saddle point

Case 4 $\det H = 0$: cannot tell (need higher order derivatives)

Example 1 Find the local min/max of $f(x,y) = x + y + \frac{1}{xy}, \quad x, y > 0$.

Step 1 *Find critical points by solving the* 2×2 *system of equations*

$$\begin{cases} f_x = 0 \\ f_y = 0 \end{cases}$$

In this case, the system is

$$\begin{cases} \frac{1}{x^2 y} = 1 \\ \frac{1}{xy^2} = 1. \end{cases}$$

Divide the first equation by the second and get $x = y$, plug back into the first equation and get $x^3 = 1$. So the only critical point is $(1,1)$.

Showed slide asking students if this point is a local max/min or saddle. Most got it right (local min). Now let's do it rigorously.

Step 2 *Compute the Hessian matrix*

$$H(x,y) = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}.$$

Recall that $f_{xy} = f_{yx}$.

In our case, get $H(x,y) = \begin{bmatrix} \frac{2}{x^3 y} & \frac{1}{x^2 y^2} \\ \frac{1}{x^2 y^2} & \frac{2}{xy^3} \end{bmatrix}$.

Step 2 *Compute the Hessian matrix at each of the critical points.*

$$H(1,1) = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}.$$

Step 4 *Apply the second derivative test for each critical point.*

$\det H(1,1) = 4 - 1 = 3 > 0$ and $f_{xx} = 2 > 0$, so $(1,1)$ is a local minimum.

Attention! We can also infer the nature of a critical point from the contour plot. Showed picture and discussed possibilities. Most students got the right answer.

Max: $f \to \infty$ when $x \to \infty$ or $y \to \infty$ or $x \to 0$ or $y \to 0$.
Min: global min at $(1,1)$ where $f(1,1) = 3$.
NOTE: the global min/max of a function is not necessarily at a critical point! Need to check boundary / infinity.
Example 2 $f(x,y) = (x^2 + y^2)e^{-x}$

Step 1 *Find critical points by solving the* 2×2 *system of equations*

$$\begin{cases} f_x = 0 \\ f_y = 0 \end{cases}$$

In this case, the system is

$$\begin{cases} (2x - x^2 - y^2)e^{-x} = 0 \\ 2ye^{-x} = 0. \end{cases}$$

The second equation tells us that $y = 0$. Plug back into the first equation and get $x^2 - 2x = 0$. So critical points are $(0, 0)$ and $(2, 0)$.

Step 2 *Compute the Hessian matrix*

$$H(x, y) = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}.$$

In our case, get $H(x, y) = \begin{bmatrix} (2 - 4x + x^2 + y^2)e^{-x} & -2ye^{-x} \\ -2ye^{-x} & 2e^{-x} \end{bmatrix}.$

Step 2 *Compute the Hessian matrix at each of the critical points.*

$$H(0, 0) = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

and

$$H(2, 0) = \begin{bmatrix} -2e^{-2} & 0 \\ 0 & 2e^{-2} \end{bmatrix}.$$

Step 4 *Apply the second derivative test for each critical point.*

- For $(0, 0)$: $\det H(0, 0) = 4 > 0$ and $f_{xx} = 2 > 0$, so $(0, 0)$ is a local minimum.
- For $(2, 0)$: $\det H(2, 0) = -4e^{-4} < 0$, so $(2, 0)$ is a saddle point.

In Example 2 above, to find the global min/max of f in the square $0 \leq x, y \leq 1$, we need to check what happens on the boundary. Namely we have to look at $f(0, y), f(1, y), f(x, 0)$ and $f(x, 1)$. We have to compute the min/max for these 4 functions and compare to the value at critical points inside the square (if any).

Values at critical points inside the square: $f(0, 0) = 0$.

Boundary:
$f(0, y) = y^2$: it has a minimum at $y = 0$
$f(1, y) = (1 + y^2)/e$: has a minimum at $y = 0$.
$f(x, 0) = x^2e^{-x}$: the first derivative is $(2x - x^2)e^{-x}$; critical points: $0, 2$; but only 0 is in our domain. Second derivative: $(2 - 4x + x^2)e^{-x}$ takes value 2 at 0. Get local min 0 at 0.
$f(x, 1) = (x^2 + 1)e^{-x}$: first derivative $(2x - x^2 - 1)e^{-x}$ is zero at $x = 1$. Second derivative Have

$f(1,1) = 2e^{-1} > 0 = f(0,0)$.

Global min $= 0$, global max $= 2e^{-1}$ in the square $[0,1] \times [0,1]$

We did not cover what follows in class, but I left in the notes as it is good practice.

Question: global min/max of $f(x,y) = (x^2 + y^2)e^{-x}$ in first quadrant, i.e. for $x, y \geq 0$.

Values at critical points: $f(0,0) = 0, f(2,0) = 4e^{-2}$.

Boundary:
$f(0,y) = y^2$: it has a minimum at $y = 0$
Fix $x > 0$: as $y \to \infty, f(x,y) \to \infty$
$f(x,0) = x^2 e^{-x}$: the first derivative is $(2x - x^2)e^{-x}$; critical points: $0, 2$; second derivative: $(2 - 4x + x^2)e^{-x}$ takes values 2 at 0 and $-2e^{-2}$ at 2. Get local min 0 at 0 and local max $4e^{-2}$ at 2.
Fix $y > 0$: as $x \to \infty, f(x,y) \to 0$.
Global min $= 0$, no global max in first quadrant.