MATH 20C Lecture 11 - Thursday, November 13, 2014

Lagrange multipliers

Problem: min/max of a function $f(x, y, z)$ when variables are constrained by an equation $g(x, y, z) = c$.

Example: find point of the hyperbola $xy = 3$ closest to origin. i.e. minimize $\sqrt{x^2 + y^2}$, or better $f(x, y) = x^2 + y^2$, subject to $g(x, y) = xy = 3$. Picture.

Observe on picture: at the minimum, the level curves are tangent to each other, so the normal vectors ∇f and ∇g are parallel.

So: there exists λ ("multiplier") such that $\nabla f = \lambda \nabla g$.

We replace the constrained min/max problem in 2 variables with 3 equations involving 3 variables x, y, λ :

$$\begin{cases} f_x = \lambda g_x \\ f_y = \lambda g_y \\ g(x, y) = c \end{cases} \qquad \text{i.e. in our case} \qquad \begin{cases} 2x = \lambda y \\ 2y = \lambda x \\ xy = 3. \end{cases}$$

Since we need $x, y \neq 0$ in the third equation, we also have $\lambda \neq 0$. Therefore we can divide in the first to get $x = \frac{\lambda y}{2}$. Replace in the second equation and get $2y = \frac{\lambda^2 y}{2}$ and so $\lambda^2 = 4 \implies \lambda = \pm 2$.

If $\lambda = 2$: then $x = y$ and the third equation yields $x = y = \pm\sqrt{3}$. Points: $\left(\sqrt{3}, \sqrt{3}\right)$ and $\left(-\sqrt{3}, -\sqrt{3}\right)$.

If $\lambda = -2$: then $x = -y$ and the third equation becomes $-x^2 = 3$ which has no solutions.

Get two points where the distance to origin is minimal, $\left(\sqrt{3}, \sqrt{3}\right)$ and $\left(-\sqrt{3}, -\sqrt{3}\right)$. For both $f\left(\sqrt{3}, \sqrt{3}\right) = f\left(-\sqrt{3}, -\sqrt{3}\right) = 6$ and the distance to origin is $\sqrt{6}$.

Warning: method doesn't say whether we have a min or a max, and second derivative test DOES NOT apply with constrained variables. Need to answer using geometric argument or by comparing values of f.

Why does this work?

We are looking for points on $g = c$ where f attains a local min/max when restricted to the level set $g = c$. This means that we want the directional partial derivative $D_{\hat{\mathbf{u}}} f = 0$ for all $\hat{\mathbf{u}}$ unit vectors tangent to $g = c$. Since $0 = D_{\hat{\mathbf{u}}} f = \nabla f \cdot \hat{\mathbf{u}}$ this means that we want ∇f to be perpendicular to the level set $g = c$. We already have a vector tangent to the level set, namely ∇g. So we want $\nabla f \parallel \nabla g$.

Example: Find the min/max of $f(x, y, z) = 3x + y + 4z$ on the surface $x^2 + 3y^2 + 6z^2 = 1$.

Step 1 *Compute the two gradients ∇f and ∇g.*

$\nabla f = \langle 3, 1, 4 \rangle \qquad \nabla g = \langle 2x, 6y, 12z \rangle$

Step 2 *Write down the Lagrange multiplier equations $\nabla f = \lambda \nabla g$ and the constraint $g = c$.*

$$\nabla f = \lambda \nabla g \quad \Longrightarrow \quad \begin{cases} 3 &= 2\lambda x \\ 1 &= 6\lambda y \\ 4 &= 12\lambda z \end{cases}$$

$$g = c \quad \Longrightarrow \quad x^2 + 3y^2 + 6z^2 = 1$$

Step 3 *Solve the system, i.e. find points (x, y, z) that satisfy the equations from Step 2.*

WARNING! There is no general method to solve these equations. In each case, you have to think about them and come up with a method. Sometimes it will be impossible to solve without using a computer. (Not on the exam though!)

In our example, note that λ cannot be 0. From the first three equations get $x = \frac{3}{2\lambda}, y = \frac{1}{6\lambda}, z = \frac{1}{3\lambda}$. Substitute these values in the constraint equation and get $\lambda^2 = 3$, so $\lambda = \pm\sqrt{3}$. Therefore there are two points $\left(\frac{\sqrt{3}}{2}, \frac{1}{6\sqrt{3}}, \frac{1}{3\sqrt{3}}\right)$ and $\left(-\frac{\sqrt{3}}{2}, -\frac{1}{6\sqrt{3}}, -\frac{1}{3\sqrt{3}}\right)$ on the given surface at which the gradients of f and g are parallel.

Step 4 *Plug the points you found into f and compare values.*

WARNING! There is no general method to tell you if the points you got are local min/max/saddle. The only way is to plug in points *on the surface/curve* $g = c$ and compare the values of f.

However, sometimes we can find some geometric reason for which the function f would be forced to have a min or a max on the surface/curve $g = c$. For instance, if $g = c$ is closed, then f will have both min and max along $g = c$.

This happens to be the case for us, since $x^2 + 3y^2 + 6z^2 = 1$ describes the shell of an ovoid in 3-space.

$$f\left(\frac{\sqrt{3}}{2}, \frac{1}{6\sqrt{3}}, \frac{1}{3\sqrt{3}}\right) = 3\frac{\sqrt{3}}{2} + \frac{1}{6\sqrt{3}} + 4\frac{1}{3\sqrt{3}} = 2\sqrt{3}.$$

$$f\left(-\frac{\sqrt{3}}{2}, -\frac{1}{6\sqrt{3}}, -\frac{1}{3\sqrt{3}}\right) = -3\frac{\sqrt{3}}{2} - \frac{1}{6\sqrt{3}} - 4\frac{1}{3\sqrt{3}} = -2\sqrt{3}.$$

The first point is a max, the second is a min.