MATH 20C Lecture 13 - Tuesday, November 25, 2014

Double integrals

Recall integral in 1-variable calculus: $\int_a^b f(x)dx =$ area below graph $y = f(x)$ over $[a, b]$.

Now: double integral $\displaystyle\iint_R f(x,y)dA =$ volume below graph $z = f(x,y)$ over region R in the xy-plane.

Cut R into small pieces $\Delta A_i \implies$ the volume is approximately $\sum f(x_i, y_i)\Delta A_i$. Limit as $\Delta A \to 0$ gives $\iint f(x,y)dA$. (demo: potato cut into french fries)

How to compute the integral? By taking slices: $S(x) =$ area of the slice by a plane parallel to yz-plane (demo: potato chips); then

$$\text{volume } = \int_{x_{\min}}^{x_{\max}} S(x)dx \quad \text{and for given } x, \quad S(x) = \int_{y_{\min}(x)}^{y_{\max}(x)} f(x,y)dy$$

BEWARE! The limits of integration in y depend on x!
In the inner integral, x is a fixed parameter, y is the integration variable. We get an *iterated integral.*
Example 1 $f(x,y) = 1 - x^2 - y^2$ and $R : 0 \le x \le 1, 0 \le y \le 1$.

$$\int_0^1 \int_0^1 \left(1 - x^2 - y^2\right) dy\, dx$$

How to evaluate?
1) inner integral (x is constant):

$$\int_0^1 \left(1 - x^2 - y^2\right) dy = \left[y - x^2 y - \frac{y^3}{3}\right]_{y=0}^{y=1} = \left(1 - x^2 - \frac{1}{3}\right) - 0 = \frac{2}{3} - x^2.$$

2)outer integral: $\displaystyle\int_0^1 \left(\frac{2}{3} - x^2\right) dx = \left[\frac{2}{3}x - \frac{x^3}{3}\right]_{x=0}^{x=1} = \frac{2}{3} - \frac{1}{3} = \frac{1}{3}.$

Note: $dA = dy\, dx = dx\, dy$, limit of $\Delta A = \Delta y \Delta x = \Delta x \Delta y$ for small rectangles.

Exchanging the order of integration

$\int_0^1\int_0^2 f(x,y)dx\, dy = \int_0^2\int_0^1 f(x,y)dy\, dx$, since region is a rectangle (drawn picture). In general, more complicated!

Example: $\displaystyle\int_0^1\int_x^{\sqrt{x}} \frac{e^y}{y} dy\, dx$ (Inner integral has no formula.)

To exchange order: 1) draw the region (here: $x \leq y \leq \sqrt{x}$ for $0 \leq x \leq 1$ – picture drawn on blackboard).

2) figure out bounds in other direction: fixing a value of y, what are the bounds for x? Here: left border is $x = y^2$, right is $x = y$; first slice is $y = 0$, last slice is $y = 1$, so we get

$$\int_0^1 \int_{y^2}^{y} \frac{e^y}{y} dx\, dy = \int_0^1 \frac{e^y}{y}(y - y^2) dy = \int_0^1 e^y(1-y) dy \overset{\text{parts}}{=} [e^y(1-y)]_{y=0}^{y=1} + \int_0^1 e^y dy = e - 2.$$

Polar coordinates

Recall: in the plane, $x = r\cos\theta, y = r\sin\theta$ where r is the distance from the origin to the (x, y) point, θ is the angle with the positive x-axis. Drawn picture.
Useful if either integrand or region have a simpler expression in polar coordinates.
Area element: $\Delta A \approx (r\Delta\theta)\Delta r$ (picture drawn of a small element with sides Δr and $r\Delta\theta$). Taking $Deltar, \Delta\theta \to 0$, we get

$$\boxed{dA = r\, dr\, d\theta.}$$

Example Integrate function $f(x, y) = 1 - x^2 - y^2$ over the quarter-disk $R : x^2 + y^2 \leq 1, 0 \leq x \leq 1, 0 \leq y \leq 1$. (computes volume between xy-plane and paraboloid in the first octant).
How to find the bounds of integration? Fix x constant and look at the slice of R parallel to y-axis. Bounds from $y = 0$ to $y = \sqrt{1 - x^2}$ in the inner integral. For the outer integral: first slice is at $x = 0$, last slice is at $x = 1$. So we get

$$\int_0^1 \int_0^{\sqrt{1-y^2}} \left(1 - x^2 - y^2\right) dx\, dy.$$

Note that the inner bounds depend on the outer variable x; the outer bounds are constants!
1) inner integral (x is constant):

$$\int_0^{\sqrt{1-x^2}} \left(1 - x^2 - y^2\right) dy = \left[(1 - x^2)y - \frac{y^3}{3}\right]_{y=0}^{y=\sqrt{1-x^2}} = (1-x^2)^{3/2} - \frac{(1-x^2)^{3/2}}{3} = \frac{2}{3}(1-x^2)^{3/2}.$$

2)outer integral:

$$\int_0^1 \frac{1}{3}(1 - x^2)^{3/2} dx = \ldots (\text{trig substitution } x = \sin\theta, \text{ double angle formulas}) \ldots = \frac{\pi}{8}.$$

This is complicated! It will be easier to do it in polar coordinates.

$$\iint_{x^2+y^2\leq 1, 0\leq x\leq 1, 0\leq y\leq 1} \left(1 - x^2 - y^2\right) dxdy = \int_0^{\pi/2} \int_0^1 (1-r^2) r\, dr\, d\theta = \int_0^{\pi/2} \left[\frac{r^2}{2} - \frac{r^4}{4}\right]_{r=0}^{r=1} d\theta = \frac{\pi}{8}.$$

Once again,

$$\boxed{\iint_R f(x,y)dA = \iint_R f(r,\theta) r\, dr\, d\theta.}$$

In general: when setting up $\iint f r\, dr\, d\theta$, find bounds as usual: given a fixed θ, find initial and final values of r (sweep region by rays).

Applications of double integrals

(Did not go over this in class due to time limitations. Might be useful for homework though.)

Computing volumes *Example:* Find the volume of the region enclosed by $z = 1 - y^2$ and $z = y^2 - 1$ for $0 \leq x \leq 2$.

Both surfaces look like parabola-shaped tunnels along the x-axis. They intersect at $1 - y^2 = y^2 - 1 \implies y = \pm 1$. So $z = 0$ and x can be anything, therefore lines parallel to the x-axis. Draw picture, please! Get volume by integrating the difference $z_{\text{top}} - z_{\text{bottom}}$, i.e. take the volume under the top surface and subtract the volume under the bottom surface (same idea as in 1 variable).

$$\begin{aligned} \pm \text{ vol } &= \int_0^2 \int_{-1}^1 \big((1-y^2) - (y^2-1)\big)\, dy\, dx = 2\int_0^2 \int_{-1}^1 (1-y^2) dy\, dx \\ &= 2\int_0^2 \left[y - \frac{y^3}{3} \right]_{y=-1}^{y=1} dx = 2\int_0^2 \frac{4}{3} dx = \frac{16}{3}. \end{aligned}$$

Since volume is always positive, our answer is 16/3.

Area of a plane region R is

$$\text{area}(R) = \iint_R 1 dA.$$

Mass the total mass of a flat object in the shape of a region R with density given by $\rho(x,y)$ is

$$\text{Mass} = \iint_R \rho(x,y) dA.$$

Average the average value of a function $f(x,y)$ over the plane region R is

$$\bar{f} = \frac{1}{\text{area}(R)} = \iint_R f(x,y) dA.$$

Weighted average of the function $f(x,y)$ over the plane region R with density $\rho(x,y)$ is

$$\frac{1}{\text{Mass}} \iint_R f(x,y)\rho(x,y) dA.$$

Center of mass of a plate with density $\rho(x, y)$ is the point with coordinates $(\bar{x}, \bar{y})$ given by weighted average

$$\bar{x} = \frac{1}{\text{Mass}} \iint_R x\rho(x, y)dA,$$

$$\bar{y} = \frac{1}{\text{Mass}} \iint_R y\rho(x, y)dA.$$