MATH 20C Lecture 14 - Tuesday, December 2, 2014

More double integrals

Example $\iint_D (x+1)y\,dA$, where $D : x \geq 0, y \geq 0, x^2 + y^2 \leq 1$.

$x = r\cos\theta, y = r\sin\theta, 0 \leq r \leq 1, 0 \leq \theta \leq \pi/2$ and the integral becomes

$$\int_0^{\pi/2} \int_0^1 (1 + r\cos\theta) r \sin\theta r\,dr\,d\theta = \int_0^{\pi/2} \int_0^1 (r^2 \sin\theta + r^3 \sin\theta\cos\theta)\,dr\,d\theta =$$

$$= \int_0^{\pi/2} \left(\frac{1}{3}\sin\theta + \frac{1}{4}\sin\theta\cos\theta\right) d\theta = \frac{1}{3}\left[-\cos\theta\right]_{\theta=0}^{\theta=\pi/2} + \frac{1}{4}\left[\frac{\sin^2\theta}{2}\right]_{\theta=0}^{\theta=\pi/2} = \frac{1}{3} + \frac{1}{8} = \frac{11}{24}.$$

Applications of double integrals

Area of a plane region R is

$$\text{area}(R) = \iint_R 1\,dA.$$

Computing volumes *Example:* Find the volume of the region enclosed by $z = 1 - y^2$ and $z = y^2 - 1$ for $0 \leq x \leq 2$.

Both surfaces look like parabola-shaped tunnels along the x-axis. They intersect at $1 - y^2 = y^2 - 1 \implies y = \pm 1$. So $z = 0$ and x can be anything, therefore lines parallel to the x-axis. Draw picture, please! Get volume by integrating the difference $z_{\text{top}} - z_{\text{bottom}}$, i.e. take the volume under the top surface and subtract the volume under the bottom surface (same idea as in 1 variable).

$$\begin{aligned} \pm\,\text{vol} &= \int_0^2 \int_{-1}^1 \left((1-y^2) - (y^2-1)\right) dy\,dx = 2\int_0^2 \int_{-1}^1 (1-y^2)\,dy\,dx \\ &= 2\int_0^2 \left[y - \frac{y^3}{3}\right]_{y=-1}^{y=1} dx = 2\int_0^2 \frac{4}{3}\,dx = \frac{16}{3}. \end{aligned}$$

Since volume is always positive, our answer is 16/3.

Mass the total mass of a flat object in the shape of a region R with density given by $\rho(x,y)$ is

$$\text{Mass} = \iint_R \rho(x,y)\,dA.$$

Average the average value of a function $f(x,y)$ over the plane region R is

$$\bar{f} = \frac{1}{\text{area}(R)} = \iint_R f(x,y)dA.$$

Weighted average of the function $f(x,y)$ over the plane region R with density $\rho(x,y)$ is

$$\frac{1}{\text{Mass}} \iint_R f(x,y)\rho(x,y)dA.$$

Center of mass of a plate with density $\rho(x,y)$ is the point with coordinates $(\bar{x},\bar{y})$ given by weighted average

$$x_{CM} = \frac{1}{\text{Mass}} \iint_R x\rho(x,y)dA,$$
$$y_{CM} = \frac{1}{\text{Mass}} \iint_R y\rho(x,y)dA.$$

Example: A plate in the shape of the region bounded by $y = x^{-1}$ and $y = 0$ for $1 \le x \le 4$ has mass density $\rho(x,y) = y/x$. Calculate the total mass of the plate.

First, draw region. Then set limits of integration.

$$\text{Mass} = \int_1^4 \int_0^{x^{-1}} \frac{y}{x} dy\, dx = \int_1^4 \left[\frac{y^2}{2x}\right]_{y=0}^{y=x^{-1}} dx = \frac{1}{2}\int_1^4 x^{-3}dx = -\frac{1}{4}\left[\frac{1}{x^2}\right]_{x=1}^{x=4} = \frac{15}{64}.$$

For the same region, center of mass has coordinates

$$x_{CM} = \frac{1}{\text{Mass}} \iint_R x\rho(x,y)dA = \frac{64}{15}\int_1^4 \int_0^{x^{-1}} y dy\, dx = \frac{64}{15}\int_1^4 \left[y^2/2\right]_{y=0}^{y=x^{-1}} dx =$$
$$= \frac{32}{15}\int_1^4 x^{-2}dx = \frac{32}{15}\left[-\frac{1}{x}\right]_{x=1}^{x=4} = \frac{8}{5}$$

and y_{CM} is left as an exercise.

Triple Integrals

$$\iiint_R f(x,y,z)\, dV \quad (R \text{ is a solid in space})$$

Note: $\Delta V = \text{area(base)} \cdot \text{height} = \Delta A \Delta z$, so $dV = dA\, dz = dx\, dy\, dz$ or any permutation of the three.
Example 1 R : the region between paraboloids $z = x^2 + y^2$ and $z = 4 - x^2 - y^2$. (picture drawn)

The volume of this region is $\iiint_R 1\,dV = \iint_D \left[\int_{x^2+y^2}^{4-x^2-y^2} dz\right] dA$, where D is the shadow in the xy-plane of the region R.

To set up bounds, (1) for fixed (x, y) find bounds for z : here lower limit is $z = x^2 + y^2$, upper limit is $z = 4 - x^2 - y^2$; (2) find the shadow of R onto the xy-plane, i.e. set of values of (x, y) above which region lies. Here: R is widest at intersection of paraboloids, which is in plane $z = 2$; general method: for which (x, y) is z on top surface $\geq z$ on bottom surface? Answer: when $4 - x^2 - y^2 \geq x^2 + y^2$, i.e. $x^2 + y^2 \leq 2$. So we integrate over a disk of radius $\sqrt{2}$ in the xy-plane. By usual method to set up double integrals, we finally get

$$\text{vol}(R) = \int_{-\sqrt{2}}^{\sqrt{2}} \int_{-\sqrt{2-x^2}}^{\sqrt{2-x^2}} \int_{x^2+y^2}^{4-x^2-y^2} dz\,dy\,dx.$$

Actual evaluation would be easier using polar coordinates:

$$\text{vol}(R) = \iint_{x^2+y^2\leq 2} (4-2x^2-2y^2)dA = \int_0^{2\pi} \int_0^{\sqrt{2}} (4-2r^2)r\,dr\,d\theta = \int_0^{2\pi} \left[\frac{1}{8}(4-2r^2)^2\right]_{r=0}^{r=\sqrt{2}} d\theta = 4\pi.$$

MATH 20C Lecture 15 - Thursday, December 4, 2014

Cylindrical coordinates

(r, θ, z) where $x = r\cos\theta, y = r\sin\theta$. (Drawn picture.) Here r measures distance from z-axis, θ measures angle from xz-plane, z is still the height.
Cylinder of radius 8 centered on z-axis is $r = 8$ (drawn); $\theta = \pi/3$ is a vertical half-plane (drawn).
Volume element: $dV = dAdz$; in cylindrical coordinates, $dA = r\,dr\,d\theta$, so

$$\boxed{dV = r\,dr\,d\theta\,dz.}$$

Once again,

$$\boxed{\iiint_R f(x, y, z)dV = \iiint_R f(r, \theta, z)r\,dr\,d\theta\,dz.}$$

Example R : portion of the half-cylinder $x^2 + y^2 \leq 4$, $x \geq 0$ such that $0 \leq z \leq 3y$. Compute the mass of the solid in the shape of R with mass-density given by $\rho(x, y, z) = z^2$.

Again, it's natural to set this up in cylindrical coordinates. The bounds for z are clear: $z_{\min} = 0$ and $z_{\max} = 3y = 3r\sin\theta$. The shadow on the xy-plane is the quarter disk $x^2 + y^2 \leq 1, x \geq 0, y \geq 0$.

$$\text{Mass}(R) = \int_0^{\pi/2} \int_0^2 \int_0^{3r\sin\theta} z^2 r\, dz\, dr\, d\theta = \int_0^{\pi/2} \int_0^2 r \left[\frac{z^3}{3}\right]_{z=0}^{z=3r\sin\theta} dr\, d\theta$$
$$= \int_0^{\pi/2} \int_0^2 9r^4 \sin^3\theta\, dr\, d\theta = 9 \int_0^{\pi/2} \frac{32}{5} \sin^3\theta\, d\theta.$$

To evaluate this last integral, write $\sin^3\theta = \sin\theta(1 - \cos^2\theta)$ and use the substitution $u = \cos\theta$. Do not forget to change the bounds of integration!

(In class I had made a mistake by letting the bounds for θ be $\pm\pi/2$.)

Applications

Volume the volume of a solid R is given by

$$\text{vol}(R) = \iiint_R 1\, dV.$$

Mass the total mass of a solid R with density given by $\rho(x, y, z)$ is

$$\text{Mass}(R) = \iiint_R \rho(x, y, z)\, dV.$$

Average the average value of a function $f(x, y, z)$ over the a solid R is

$$\bar{f} = \frac{1}{\text{vol}(R)} = \iiint_R f(x, y, z)\, dV.$$

Weighted average of the function $f(x, y, z)$ over the solid R with density $\rho(x, y, z)$ is

$$\frac{1}{\text{Mass}(R)} \iiint_R f(x, y, z)\rho(x, y, z)\, dV.$$

Center of mass of a solid with density $\rho(x, y, z)$ is the point with coordinates (x_{CM}, y_{CM}, z_{CM}) given by weighted average

$$x_{CM} = \frac{1}{\text{Mass}(R)} \iiint_R x\rho(x, y, z)\, dV,$$

$$y_{CM} = \frac{1}{\text{Mass}(R)} \iiint_R y\rho(x, y, z)\, dV,$$

$$z_{CM} = \frac{1}{\text{Mass}(R)} \iiint_R z\rho(x, y, z)\, dV.$$

Example: Let R be a solid in the shape of the first octant of the unit ball. Assume the density is given by $\rho(x,y,z) = y$. Find the z-coordinate of the center of mass of R.

Solution First drawn picture of R. The unit sphere has equation $x^2+y^2+z^2 = 1$. It intersects the xy-plane in the unit circle $x^2+y^2 = 1$. We want only the parts with $x \geq 0, y \geq 0, z \geq 0$. Need to compute Mass($R$) and

$$\iiint_R z\rho(x,y,z)\,dV = \iint_{\text{quarter unit disk}} \left[\int_0^{\sqrt{1-x^2-y^2}} yz\,dz\right] dA$$

$$= \int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} yz\,dz\,dy\,dx = \int_0^1 \int_0^{\sqrt{1-x^2}} \left[y\frac{z^2}{2}\right]_{z=0}^{z=\sqrt{1-x^2-y^2}} dy\,dx = \dots$$

Example (based on Example 5 from Section 15.3 in the book) Let R be the region in the first octant (i.e. $x \geq 0, y \geq 0, z \geq 0$) bounded by

$$z = 4 - y^2, \quad y = 2x, \quad z = 0, \quad x = 0.$$

Consider $f(x,y,z) = xyz$. Set up the integral $\iiint_R f(x,y,z)\,dV$, in two different ways.

Showed picture of R and its projections on the various coordinate planes.

Setup #1 Integrate first with respect to z, then with respect to y.

The surface $z = 4 - y^2$ intersects the first quadrant of the xy-plane in the line $y = 2$. The projection of the xy-plane is a triangle bounded by the y-axis and the lines $y = 2$ and $y = 2x$. For each point (x,y) the vertical segment above it goes from $z = 0$ to $z = 4 - y^2$. Get

$$\int_0^1 \int_{2x}^2 \int_0^{4-y^2} xyz\,dz\,dy\,dx.$$

Setup #2 Integrate first with respect to x, then with respect to z.

The projection onto the yz-plane is bounded by the y-axis, the z-axis and the parabola $z = 4 - y^2$. For each point (y,z) the segment in the direction of the x-axis goes from $x = 0$ to $x = y/2$. Get

$$\int_0^2 \int_0^{4-y^2} \int_0^{y/2} xyz\,dx\,dz\,dy.$$

Calculate:

$$\int_0^2 \int_0^{4-y^2} \int_0^{y/2} xyz\,dx\,dz\,dy = \int_0^2 \int_0^{4-y^2} \left[\frac{x^2yz}{2}\right]_{x=0}^{x=y/2} dz\,dy = \frac{1}{8}\int_0^2 \int_0^{4-y^2} y^3z\,dz\,dy = \frac{1}{16}\int_0^2 y^3(4-y^2)^2 dy$$

$$= \frac{1}{16}\int_0^2 (16y^3 - 8y^5 + y^7)dy = \frac{1}{16}\left[4y^4 - \frac{4}{3}y^6 + \frac{1}{8}y^8\right]_{y=0}^{y=2} = \frac{1}{16}\left(64 - \frac{256}{3} + 32\right) = 6 - \frac{16}{3} = \frac{2}{3}.$$

The other way,

$$\int_0^1 \int_{2x}^2 \int_0^{4-y^2} xyz\, dz\, dy\, dx = \int_0^1 \int_{2x}^2 \left[\frac{xyz^2}{2} \right]_{z=0}^{z=4-y^2} dy\, dx = \frac{1}{2} \int_0^1 \int_{2x}^2 xy(4-y^2)^2\, dy\, dx =$$

$$= \frac{1}{2} \int_0^1 \left[\frac{x(4-y^2)^3}{6} \right]_{y=2x}^{y=2} = -\frac{1}{12} \int_0^1 x(4-4x^2)^3 dx = -\frac{1}{12} \left[\frac{(4-4x^2)^4}{32} \right]_{x=0}^{x=1} = \frac{2^8}{2^7 \cdot 3} = \frac{2}{3}.$$