Mathematics 20D
Final Exam, December 18, 2014

Maximum score on this exam is 100 points.

Write your solutions in the space provided. In case need more space than provided use the back-page.

Important: Show your reasoning! Answers without explanations will receive no credit, even if they are correct!

Full name: ____________

Section: ____________

Student ID: ____________

Signature: ____________

Problem 1. ________

Problem 2. ________

Problem 3. ________

Problem 4. ________

Problem 5. ________

Problem 6. ________

Problem 7. ________

Problem 8. ________

Total: ____________

Problem 1. (8 points) Solve the following differential equation:

$$y^{(4)} - 4y = 0, y(0) = 1, y'(0) = 0, y''(0) = -2, y'''(0) = 0.$$

Problem 2. (10 points) In the problem below you **MUST** use the Laplace transform method. No credit is given otherwise!

Solve the following initial data problem

$$y'' - 2y' + y = f(t), \quad y(0) = 2, y'(0) = 1$$

where

$$f(t) = \begin{cases} t, & 0 < t < 2 \\ -1, & 2 \leq t. \end{cases}$$

Problem 3. (14 points) In the problem below you **MUST NOT** use the Laplace transform method. No credit is given otherwise! Also you **NEED** to show reasoning; guessing the solution will not receive full credit.

i) Solve the following initial data problem

$$y'' - 2y' + y = t, \quad y(0) = 2, y'(0) = 1.$$

Compute $a = y(2), b = y'(2)$.

ii) Solve the following initial data problem

$$y'' - 2y' + y = -1, \quad y(2) = a, y'(2) = b.$$

where a, b were obtained in part i).

iii) Compare the solutions obtained here with the ones from Problem 2. Interpret the result.

Problem 4. (8 points) Find the general solution of the following equation:

$$(e^x \cos y + 2\cos x)y' + e^x \sin y - 2y \sin x = 0.$$

Problem 5. (16 points) Consider the following equation

$$(3 - x^2)y'' - 3xy' - y = 0$$

Derive the general solution as a power series at $x_0 = 0$ by performing the following steps:

i) write down the recursion formula.

ii) compute the first 6 terms in the series (this should go up to the term containing x^5).

iii) identify the two fundamental solutions, y_1 and y_2, as power series. Find a pattern in the expressions giving the coefficients of y_1, y_2.

iv) what is the radius of convergence of the general solution?

Problem 6. (20 points) Consider the following system:

$$\mathbf{x}' = \begin{pmatrix} 1 & 2 \\ -5 & -1 \end{pmatrix} \mathbf{x}. \tag{1}$$

i) Find the general solution of the system.
ii) Find a fundamental matrix $\Psi(t)$ and compute its inverse $\Psi^{-1}(t)$.
iii) Find e^{At} for

$$A = \begin{pmatrix} 1 & 2 \\ -5 & -1 \end{pmatrix}.$$

iv) Find the general solution to the nonhomogeneous system

$$\mathbf{x}' = \begin{pmatrix} 1 & 2 \\ -5 & -1 \end{pmatrix} \mathbf{x} + \begin{pmatrix} -2t \\ 1+t \end{pmatrix}.$$

v) The trajectories of the homogeneous system (1) are ellipses centered at the origin. Prove that this is indeed the case - it is enough to do it for one of the fundamental solutions you found above. To be more precise, if $x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$ is a solution to (1), show that there are a, b, c with $a, c > 0, ac > b^2$ such that

$$ax_1^2 + 2bx_1x_2 + cx_2^2 = 1.$$

Problem 7. (10 points) Solve the following differential equation:

$$y' = \frac{x^2 + 3y^2}{2xy}.$$

HINT: this is a homogeneous equation; use the substitution $y(x) = x \cdot v(x)$, derive the equation for v and solve that equation instead.

Problem 8. (14 points) John H. is 25 years old, starts his job with a salary of $80,000$ per year. His salary is indexed (increases) at a continuous rate of 3% per year. He puts 5% of his salary in his $401K$ account and his employer matches that amount. His retirement account returns 5% per year.

i) Write a differential equation (or system), including initial condition(s), modeling his $401K$ account.

ii) How much money does John H. have in his $401K$ at the age of 65? Since you are not allowed to use calculators it is fine to have a final answer containing powers of e.

iii) Assuming John H. retires at the age of 65 and plans to spend $100,000$ per year from his $401K$ account, for how long can he rely on that account? The final answer may contain powers of e, logarithms and the number you found in ii). It is assumed that the return of his account is the same 5%.

Use this sheet for extra space.