Name: SOLUTIONS

PID: ______

TA: ______

Section #: ______ Section time: ______

Seat: ______

There are 6 pages and 4 questions, for a total of 100 points.

No calculators, no electronic devices, no books, no notes, except for the one 8.5in×11in sheet of notes.

Please turn off all electronic devices.

Answer the questions in the spaces provided on the question sheets. Unless otherwise stated, show all your work for full credit. To maximize credit, cross out incorrect work.

Good luck! ☺

Question:	1	2	3	4	Total
Points:	20	25	25	30	100
Score:					

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Section #: Section time:

1. (a) (15 points) Find all the solutions of the differential equation

$$\frac{dy}{dx} = \frac{2x(y-1)^2}{x^2+1}.$$

$$\int \frac{dy}{(y-1)^2} = \int \frac{2x}{x^2+1}\,dx \qquad \text{or} \qquad y-1=0$$

$$\frac{-1}{(y-1)} = \ln|x^2+1| + C \qquad \text{or} \qquad y=1$$

$$y-1 = \frac{-1}{\ln|x^2+1|+C}$$

$$y = 1 - \frac{1}{\ln(x^2+1)+C} \qquad (\text{as } x^2+1>0)$$

(b) (5 points) Solve the initial value problem with $y(0) = 1$.

$$y = 1$$

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Section #: Section time:

2. (a) (10 points) Show that the following equation is exact:

$$\underbrace{2x + 3y^2}_{M} + \underbrace{(6xy + \cos y)}_{N}y' = 0.$$

Check $M_y = N_x$:

$M_y = 6y$

$N_x = 6y$ ✓

Yes, it is exact ✓

(b) (15 points) Find the general solution of this differential equation. *It suffices to give the solution implicitly.*

Want $\varphi(x,y)$ so that $\varphi_x = M$, $\varphi_y = N$

$\int M\,dx = x^2 + 3y^2x$

$\int N\,dy = 3xy^2 + \sin y$

$\Rightarrow \varphi(x,y) = 3xy^2 + \sin y + x^2$

Sol'n: $\boxed{3xy^2 + \sin y + x^2 = c}$

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3. (25 points) Solve the initial value problem

$$y'' + 2y + 2y = 0, \qquad y(0) = 2, \quad y'(0) = -1.$$

$$r^2 + 2r + 2 = 0 \implies r_{12} = \frac{-2 \pm 2i}{2} = -1 \pm i$$

$$\implies y(t) = c_1 e^{-t} \cos t + c_2 e^{-t} \sin t$$

$$\implies y'(t) = -e^{-t}(c_1 \cos t + c_2 \sin t) -$$

$$- c_1 e^{-t} \sin t + c_2 e^{-t} \cos t$$

$$y(0) = 2 \implies c_1 = 2$$

$$y'(0) = -1 \implies -c_1 + c_2 = -1 \implies c_2 = 1$$

$$\implies y(t) = e^{-t}(2 \cos t + \sin t)$$

4. (a) (10 points) A tank holds $V = 1000$ liters of salt water with concentration $c = 50$ grams/liter. Suppose that fresh water is added at the rate of $r = 2$ liters/hour. A mixer keeps the salt essentially uniformly distributed in the tank. A pipe lets solution out of the tank at the same rate of 2 liters/hour. Write down the differential equation for the *amount* of salt in the tank (not the concentration!) and justify your answer. Use the notation $y(t)$ for the numbers of **kilograms** of salt in the tank at time t. *Check the units in your equation!*

$y(t)$ = salt in tank (in kg) at time t

$$\frac{dy}{dt} = \text{salt in}/\text{hour} - \text{salt out}/\text{hour}$$

$$\frac{dy}{dt} = \underbrace{0}_{\text{fresh water in, so no salt gets added}} - \underbrace{2}_{\text{rate of water out of tank } \ell/hr} \cdot \underbrace{\frac{y}{V}}_{kg/\ell \text{ of salt}}$$

kg/ℓ of salt $= \frac{y}{V} = \frac{y}{1000 \text{ liters}}$

units $\quad kg/hr = \ell/hr \cdot kg/\ell \quad$ ✓

equation
$$\frac{dy}{dt} = -\frac{2y}{1000}$$

$$\frac{dy}{dt} = -\frac{y}{500}$$

(b) (15 points) Solve the equation.

$$\frac{dy}{dt} = -\frac{y}{500}$$

$$y(t) = c\, e^{-\frac{t}{500}} \quad \text{directly}$$

or

$y = 0$

$y \neq 0$

$$\frac{dy}{y} = -\frac{dt}{500}$$

$$\int \frac{dy}{y} = \int -\frac{dt}{500}$$

$$\ln|y| = -\frac{1}{500} \ln|t| + C_1$$

$$y = c\, e^{-\frac{t}{500}}$$

$$y(0) = 50 \cdot 1000 \text{ grams} = 50\text{kg}$$

$$\Rightarrow C = 50$$

$$\boxed{y = 50\, e^{-\frac{t}{500}}}$$

(c) (5 points) Assuming this process continues indefinitely, what amount of salt will be left in the tank? Does your result jibe with simple logic?

as $t \to \infty$, $y(t) = 50\, e^{-\frac{t}{500}} \to 0$, so no salt left.

Since we keep adding fresh water and draining salt, it is natural that we would be left with no salt in the tank.