Name: SOLUTIONS

PID:

TA:

Section #: ________ Section time: ________

Seat:

There are 6 pages and 4 questions, for a total of 100 points.

No calculators, no electronic devices, no books, no notes, except for the one 8.5in×11in sheet of notes.

Please turn off all electronic devices.

Answer the questions in the spaces provided on the question sheets. Unless otherwise stated, show all your work for full credit. To maximize credit, cross out incorrect work.

Good luck! ☺

Question:	1	2	3	4	Total
Points:	20	25	25	30	100
Score:					

1. (a) (15 points) Find all the solutions of the differential equation

$$\frac{dy}{dx} = \frac{2x(y+1)^2}{x^2+1}.$$

$$\int \frac{dy}{(y+1)^2} = \int \frac{2x}{x^2+1}\,dx \quad \text{or} \quad y+1=0$$

$$\frac{-1}{y+1} = \ln|x^2+1| + C \quad \text{or} \quad y=-1$$

$$y+1 = \frac{-1}{\ln|x^2+1|+C}$$

$$y = \frac{1}{C-\ln(x^2+1)} - 1 \qquad (\text{as } x^2+1>0)$$

(b) (5 points) Solve the initial value problem with $y(0) = -1$.

$$y=-1$$

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2. (a) (10 points) Show that the following equation is exact:

$$\underbrace{\sin x + 2xy^2}_{M} + (\underbrace{2x^2y - 8y^3}_{N})y' = 0.$$

Check $M_y = N_x$:

$M_y = 4xy$

$N_x = 4xy$

Yes, it is exact ✓

(b) (15 points) Find the general solution of this differential equation. *It suffices to give the solution implicitly.*

Want $\Psi(x,y)$ so that $\Psi_x = M$ and $\Psi_y = N$

$\int M\,dx = -\cos x + x^2y^2$

$\int N\,dy = x^2y^2 - 2y^4$

$\Rightarrow \Psi(x,y) = x^2y^2 - \cos x - 2y^4$

Sol'n: $\boxed{x^2y^2 - \cos x - 2y^4 = c}$

3. (25 points) Solve the initial value problem

$$y'' - 4y' + 8y = 0, \qquad y(0) = 1, \quad y'(0) = 6.$$

$$r^2 - 4r + 8 = 0 \Rightarrow r_{1,2} = \frac{4 \pm \sqrt{16-32}}{2}$$

$$\Rightarrow r_{1,2} = 2 \pm i2$$

$$\Rightarrow y(t) = e^{2t}(c_1 \cos 2t + c_2 \sin 2t)$$

$$\Rightarrow y'(t) = 2e^{2t}(c_1 \cos 2t + c_2 \sin 2t) +$$

$$+ e^{2t}(-2c_1 \sin 2t + 2c_2 \cos 2t)$$

$$y(0) = 1 \Rightarrow c_1 = 1$$

$$y'(0) = 6 \Rightarrow 2c_1 + 2c_2 = 6 \Rightarrow c_2 = 2$$

$$\Rightarrow y(t) = e^{2t}(\cos 2t + 2 \sin 2t)$$

4. (a) (10 points) A tank holds $V = 2000$ liters of salt water. Suppose that a saline solution with with concentration $c = 3$ grams/liter is added at the rate of $r = 5$ liters/hour. A mixer keeps the salt essentially uniformly distributed in the tank. A pipe lets solution out of the tank at the same rate of 5 liters/hour. Write down the differential equation for the *amount* of salt in the tank (not the concentration!) and justify your answer. Use the notation $y(t)$ for the numbers of grams of salt in the tank at time t. *Check the units in your equation!*

$y(t)$ = salt in tank at time t (in grams)

$$\underbrace{\frac{dy}{dt}}_{gr/hr} = \text{salt in}/hr - \text{salt out}/hr$$

Salt in : $3\ gr/liter \cdot 5\ liter/hr = 15\ gr/hr$

Salt out : $5\ liter/hr \cdot \underbrace{\frac{y(t)}{V}}_{gr/liter} = \cancel{\frac{5y}{2000}}$

$$= \frac{5y}{2000}\ gr/hr = \frac{y}{400}\ gr/hr$$

Equation : $\boxed{\frac{dy}{dt} = 15 - \frac{y}{400}}$

(b) (15 points) Solve the equation under the assumption that $y(0) = 0$ (i.e. initially the tank contains fresh water).

$$\frac{dy}{dt} = 15 - \frac{y}{400} \qquad y(0) = 0$$

$$\frac{dy}{dt} = \frac{6000 - y}{400}$$

$$\frac{dy}{y - 6000} = -\frac{dt}{400}$$

$$y - 6000 = c\, e^{-\frac{t}{400}}$$

$$y = 6000 + c\, e^{-\frac{t}{400}}$$

$$y(0) = 0 \Rightarrow 6000 + c = 0 \Rightarrow c = -6000$$

$$y = 6000 - 6000\, e^{-\frac{t}{400}} = 6000\left(1 - e^{-\frac{t}{400}}\right)$$

(c) (5 points) What is the limiting amount of salt in the tank? Does your result jibe with simple logic?

As $t \to \infty$, $y = 6000\left(1 - e^{-\frac{t}{400}}\right) \longrightarrow 6000$

Simple logic dictates in the end the tank will contain $3g/\text{liter}$ salt, i.e. $3 \cdot 2000 = 6000$ grams

The two are compatible.