MATH 20D
Exam 1 v6

Lecture A
April 21, 2015

Name: SOLUTIONS

PID: _____

TA: _____

Section #: _____ Section time: _____

Seat: _____

There are 6 pages and 4 questions, for a total of 100 points.

No calculators, no electronic devices, no books, no notes, except for the one 8.5in×11in sheet of notes.

Please turn off all electronic devices.

Answer the questions in the spaces provided on the question sheets. Unless otherwise stated, show all your work for full credit. To maximize credit, cross out incorrect work.

Good luck! ☺

Question:	1	2	3	4	Total
Points:	20	25	25	30	100
Score:					

1. (a) (15 points) Find all the solutions of the differential equation

$$\frac{dy}{dx} = 2x(y+1) - 1 - y.$$

$$= (2x-1)(y+1)$$

$$\int \frac{dy}{y+1} = \int (2x-1)\,dx \quad \text{or} \quad y+1 = 0$$

$$\ln|y+1| = x^2 - x + C \quad \text{or} \quad y = -1$$

$$|y+1| = e^{x^2-x+C}$$

$$y+1 = Ae^{x^2-x}, \quad A \neq 0$$

$$y = Ae^{x^2-x} - 1$$

(b) (5 points) Solve the initial value problem with $y(1) = 10$.

$$10 = Ae^0 - 1 = A - 1$$

$$A = 11$$

$$y = 11e^{x^2-x} - 1$$

2. (a) (10 points) Show that the following equation is exact:

$$\underbrace{\frac{1}{x-1} - 3x^2y^2}_{M} + \underbrace{(3y^2 - 2x^3y)y'}_{N} = 0, \quad x > 1.$$

Check $M_y = N_x$:

$M_y = -3x^2(2y) = -6x^2y$

$N_x = -2(3x^2)y = -6x^2y$

Yes, it is exact ✓

(b) (15 points) Find the general solution of this differential equation. *It suffices to give the solution implicitly.*

Want $\psi(x,y)$ so that $\psi_x = M$, $\psi_y = N$

$\int M\,dx = \ln|x-1| - x^3y^2$

$\int N\,dy = y^3 - x^3y^2$

$\Rightarrow \psi(x,y) = y^3 + \ln|x-1| - x^3y^2$

Sol'n: $\boxed{y^3 + \ln|x-1| - x^3y^2 = c}$

3. (25 points) Solve the initial value problem

$$5y'' + 6y' + y = 0, \qquad y(-1) = 5e^{1/5} - 2e, \quad y'(-1) = -e^{1/5} + 2e.$$

$5r^2 + 6r + 1 = 0 \Rightarrow r_{1,2} = \dfrac{-6 \pm \sqrt{36 - 20}}{10}$

$\Rightarrow r_1 = -1, \quad r_2 = -\frac{1}{5}$

$\Rightarrow y(t) = c_1 e^{-t} + c_2 e^{-\frac{1}{5}t} \Rightarrow y'(t) = -c_1 e^{-t} - \frac{c_2}{5} e^{-\frac{t}{5}}$

$y(-1) = 5e^{+1/5} - 2e \Rightarrow c_1 e + c_2 e^{1/5} = 5e^{1/5} - 2e$

$y'(-1) = -e^{1/5} + 2e \Rightarrow -c_1 e - \frac{c_2}{5} e^{1/5} = -e^{1/5} + 2e$

(+)

$\Rightarrow c_2 = 5$

$c_1 = -2$

$\Rightarrow y(t) = -2e^{-t} + 5e^{-\frac{1}{5}t}$

4. Mr. and Mrs. Smith want to plan for their retirement. They estimate that their retirement portfolio will generate an interest 2% a year. With mortgage and all debt cleared, they estimate that their living expenses will be \$60,000 a year, and they would like to have another \$10,000 a year as income for holidays and emergencies.

   (a) (10 points) Write down a differential equation that models the amount of money in the savings account assuming they spend the extra \$10K/year in addition to their living expenses. Use $x(t)$ to denote the balance of the account measured in thousands of dollars after t years.

$\frac{dx}{dt}$ = income − expenses (~~per~~ year)

thousand\$/year

income: comes from interest, $\frac{2}{100}x = \frac{x}{50}$ thousand \$/year

expenses: 60 + 10 (thousands of dollars)/year = 70

$$\frac{dx}{dt} = \frac{x}{50} - 70$$

   (b) (10 points) Solve the equation.

$$\frac{dx}{dt} = \frac{x-3500}{50}$$

$$\frac{dx}{x-3500} = \frac{dt}{50}$$

$$x - 3500 = ce^{\frac{t}{50}} \Rightarrow x = 3500 + ce^{\frac{t}{50}}$$

(c) (10 points) What amount of money do they need to have saved when they retire in order to keep their capital intact? Is that a stable, unstable or semistable equilibrium? Explain why.

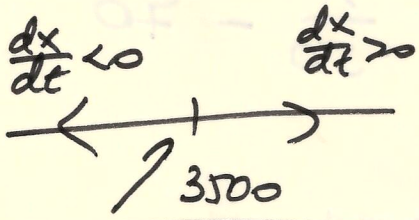
$x = 3500 + C e^{\frac{t}{50}}$: decreases if $C<0$
increases if $C>0$
~~stable~~ constant if $C=0$

So they need 3500 thousands \$ =
= 3.5 million dollars
(at least)

$\frac{dx}{dt} = \frac{x-3500}{50}$ autonomous equation

equilibrium: $x - 3500 = 0$
$x = 3500$

phase line



unstable equilibrium