

Name: = SOLUTIONS

PID: _____

TA: _____

Section #: _____ Section time: _____

Seat: _____

There are 6 pages and 3 questions, for a total of 100 points.

No calculators, no electronic devices, no books, no notes, except for the one 8.5in×11in sheet of notes. No other assistance is permitted during this exam.

Please turn off and put away your cellphone and all other electronic devices.

Answer the questions in the spaces provided on the question sheets.

Read each question carefully, and answer each question completely.

Answer the questions in the spaces provided on the question sheets.

Show all of your work; no credit will be given for unsupported answers.

Write your solutions clearly and legibly; no credit will be given for illegible solutions.

To maximize credit, cross out incorrect work.

If any question is not clear, ask for clarification.

Good luck! ☺

Question:	1	2	3	Total
Points:	40	15	45	100
Score:				

1. (a) (15 points) Find the general solution to the second order homogeneous equation

$$y'' + 2y' + y = 0.$$

$$r^2 + 2r + 1 = 0$$
$$= (r+1)^2$$

$r = -1$ so $y = e^{-t}$ is a solution

repeated root so $y = te^{-t}$ is another solution

$y = C_1 e^{-t} + C_2 t e^{-t}$ is the general solution

- (b) (25 points) Use the method of undetermined coefficients to find the general solution of the second order nonhomogeneous differential equation

$$y'' + 2y' + y = 5t + e^{-2t}$$

$\underbrace{\quad}_{y_1} \quad \underbrace{\quad}_{y_2}$

Try solution of form:

$$y_1(t) = At + B$$

$$y_1'(t) = A$$

$$y_1''(t) = 0$$

Plug in:

$$y_1'' + 2y_1' + y_1 = 5t$$

$$0 + 2A + At + B = 5t$$

$$A = 5 \quad 2A + B = 0$$

$$\Rightarrow B = -10$$

$$y_2(t) = Ce^{-2t}$$

$$y_2'(t) = -2Ce^{-2t}$$

$$y_2''(t) = 4Ce^{-2t}$$

Plug in

$$y_2'' + 2y_2' + y_2 = e^{-2t}$$

$$(4C - 4C + C)e^{-2t} = e^{-2t}$$

$$C = 1$$

$$\text{so } Y_p(t) = y_1(t) + y_2(t) = 5t - 10 + e^{-2t}$$

General solution:

$$y(t) = C_1 e^{-t} + C_2 t e^{-t} + 5t - 10 + e^{-2t}$$

2. (15 points) Determine whether the following vectors are linearly independent or not.

$$\vec{u} \quad \vec{v}$$
$$\begin{pmatrix} -1 \\ 2 \end{pmatrix} \text{ and } \begin{pmatrix} 2 \\ -4 \end{pmatrix}$$

Sol I

$$c_1 \vec{u} + c_2 \vec{v} = \vec{0}$$

$$c_1 \begin{pmatrix} -1 \\ 2 \end{pmatrix} + c_2 \begin{pmatrix} 2 \\ -4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{aligned} -c_1 + 2c_2 &= 0 \Rightarrow c_1 = 2c_2 \\ 2c_1 - 4c_2 &= 0 \Rightarrow c_1 = 2c_2 \end{aligned}$$

For instance $c_1 = 2, c_2 = 1$ is a solution.

Indeed $2 \begin{pmatrix} -1 \\ 2 \end{pmatrix} + 1 \cdot \begin{pmatrix} 2 \\ -4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ so vectors are linearly dependent.
(not lin. indep.)

Sol II

notice that $-2\vec{u} = -2 \begin{pmatrix} -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ -4 \end{pmatrix} = \vec{v}$

Since \vec{v} is a ^{nonzero} multiple of \vec{u} , the vectors are lin. dependent

Sol III

$$\det(\vec{u} : \vec{v}) = \det \begin{pmatrix} -1 & 2 \\ 2 & -4 \end{pmatrix} = (-1)(-4) - 2 \cdot 2 = 0$$

\Downarrow
vectors are lin. dependent

3. Consider the system of differential equations

$$\begin{cases} x' = -x - y \\ y' = x - y. \end{cases}$$

(a) (20 points) Find the general solution.

$$A = \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix} \quad \det(A - \lambda I) = \begin{vmatrix} -1-\lambda & -1 \\ 1 & -1-\lambda \end{vmatrix} = (1+\lambda)^2 + 1$$

$$\text{for } \lambda = -1 + i \quad \Rightarrow \lambda = -1 \pm i$$

$$\begin{pmatrix} -i & -1 \\ 1 & -i \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \mathbf{0} \Rightarrow -iu_1 - u_2 = 0$$

$$\Rightarrow u_2 = -iu_1$$

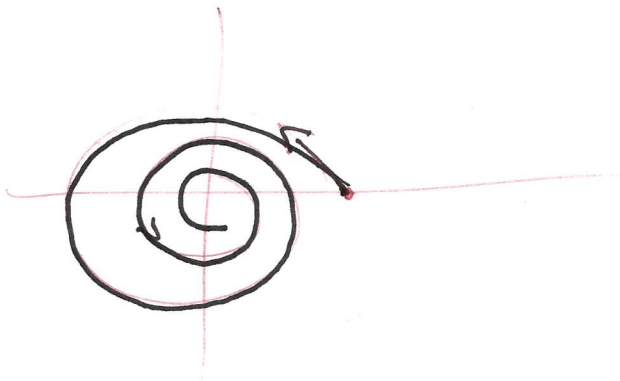
$$\Rightarrow \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = u_1 \begin{pmatrix} 1 \\ -i \end{pmatrix}$$

$$\begin{aligned} \Rightarrow \tilde{x}_1(t) &= e^{(-1+i)t} \begin{pmatrix} 1 \\ -i \end{pmatrix} = e^{-t} (\cos t + i \sin t) \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} + i \begin{pmatrix} 0 \\ -1 \end{pmatrix} \right) \\ &= e^{-t} \left[\begin{pmatrix} \cos t \\ \sin t \end{pmatrix} + i \begin{pmatrix} \sin t \\ -\cos t \end{pmatrix} \right] \end{aligned}$$

$$\Rightarrow \tilde{x}(t) = c_1 e^{-t} \begin{pmatrix} \cos t \\ \sin t \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} \sin t \\ -\cos t \end{pmatrix}$$

(b) (15 points) Sketch the phase portrait. Summarize the main features and explain why it looks that way. Specify what kind of point the origin is (e.g. spiral sink, nodal source, etc.)

$$\lambda = -1 \pm i \quad \operatorname{Re}(\lambda) < 0 \quad \Rightarrow \quad \text{stable spiral}$$
$$\begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$



(c) (10 points) Find the solution with initial condition $x(0) = 5, y(0) = 1$.

$$\begin{pmatrix} 5 \\ 1 \end{pmatrix} = \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} c_1 \\ -c_2 \end{pmatrix} \Rightarrow \begin{matrix} c_1 = 5 \\ c_2 = -1 \end{matrix}$$
$$\Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = 5 e^{-t} \begin{pmatrix} \cos t \\ \sin t \end{pmatrix} = e^{-t} \begin{pmatrix} 5 \cos t \\ -\sin t \end{pmatrix}$$