MATH 20D

Exam 2 v4

Lecture A

19 May 2015

Name: = SOLUTIONS=

PID: __________

TA: __________

Section #: ______ Section time: __________

Seat: __________

There are 6 pages and 3 questions, for a total of 100 points.

No calculators, no electronic devices, no books, no notes, except for the one 8.5in×11in sheet of notes. No other assistance is permitted during this exam.

Please turn off and put away your cellphone and all other electronic devices.

Answer the questions in the spaces provided on the question sheets.
Read each question carefully, and answer each question completely.
Answer the questions in the spaces provided on the question sheets.
Show all of your work; no credit will be given for unsupported answers.
Write your solutions clearly and legibly; no credit will be given for illegible solutions.
To maximize credit, cross out incorrect work.
If any question is not clear, ask for clarification.

Good luck! ☺

Question:	1	2	3	Total
Points:	40	15	45	100
Score:				

Name:

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1. (a) (15 points) Find the general solution to the second order homogeneous equation

$$y'' - 4y' + 4y = 0.$$

$r^2 - 4r + 4 = 0$

$= (r-2)^2$

$r = 2$ so $y = e^{2t}$ is a solution

repeated root so $y = te^{2t}$ is another solution

$y = C_1 e^{2t} + C_2 t e^{2t}$ is the general solution

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(b) (25 points) Use the method of undetermined coefficients to find the general solution of the second order nonhomogeneous differential equation

$$y'' - 4y' + 4y = \underbrace{3t}_{y_1} - \underbrace{e^{-t}}_{y_2}.$$

Try particular solution of the form:

$y_1(t) = At + B$

$y_1'(t) = A$

$y_1''(t) = 0$

Plug in:

$y_1'' - 4y_1' + 4y_1 = 3t$

$-4A + 4(At+B) = 3t$

$4A = 3 \qquad -4A + 4B = 0$

$A = 3/4 \qquad B = 3/4$

$y_2(t) = Ce^{-t}$

$y_2'(t) = -Ce^{-t}$

$y_2''(t) = Ce^{-t}$

Plug in:

$y_2'' - 4y_2' + 4y_2 = -e^{-t}$

$(C + 4C + 4C)e^{-t} = -e^{-t}$

$9C = -1 \quad C = -1/9$

So particular solution is:

$y_p(t) = y_1 + y_2 = 3/4\, t + 3/4 - 1/9\, e^{-t}$

General solution:

$$\boxed{y(t) = c_1 e^{2t} + c_2 t e^{2t} + 3/4\, t + 3/4 - 1/9\, e^{-t}}$$

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2. (15 points) Determine whether the following vectors are linearly independent or not.

$$\overset{\vec{u}}{\begin{pmatrix} 2 \\ 4 \end{pmatrix}} \text{ and } \overset{\vec{v}}{\begin{pmatrix} -3 \\ -6 \end{pmatrix}}$$

Sol I $\quad c_1 \vec{u} + c_2 \vec{v} = \vec{0}$

$$c_1 \begin{pmatrix} 2 \\ 4 \end{pmatrix} + c_2 \begin{pmatrix} -3 \\ -6 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} 2c_1 - 3c_2 = 0 & \Rightarrow c_1 = \frac{3}{2} c_2 \\ 4c_1 - 6c_2 = 0 & \Rightarrow c_1 = \frac{3}{2} c_2 \end{cases}$$

Hence $c_1 = 3$, $c_2 = 2$ is a solution

Indeed $3 \begin{pmatrix} 2 \\ 4 \end{pmatrix} + 2 \begin{pmatrix} -3 \\ -6 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \vec{u}, \vec{v}$ are linearly dependent (not lin. indep)

Sol II Notice that $\begin{pmatrix} 2 \\ 4 \end{pmatrix} = -\frac{2}{3} \cdot \begin{pmatrix} -3 \\ -6 \end{pmatrix}$

Since $\vec{u}$ is a nonzero multiple of $\vec{v}$, the vectors are linearly dependent.

Sol III $\quad \det(\vec{u} : \vec{v}) = \det \begin{pmatrix} 2 & -3 \\ 4 & -6 \end{pmatrix} = 2 \cdot (-6) - 4(-3) = 0$

Since $\det(\vec{u} : \vec{v}) = 0$, the vectors are linearly dependent

3. Consider the system of differential equations

$$\begin{cases} x' = x - y \\ y' = x + y. \end{cases}$$

(a) (20 points) Find the general solution.

$A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$ $\det(A - \lambda I_2) = (1-\lambda)(1-\lambda) + 1 =$

$= \lambda^2 - 2\lambda + 2$

$\Rightarrow$ the eigenvalues are $\lambda_{1,2} = 1 \pm i$

Eigenvector for $\lambda = 1 + i$:

$\begin{pmatrix} -i & -1 \\ 1 & -i \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow -ia - b = 0 \Rightarrow b = -ia$

$\Rightarrow \begin{pmatrix} a \\ b \end{pmatrix} = a\begin{pmatrix} 1 \\ -i \end{pmatrix}$

$\tilde{x}_1(t) = \begin{pmatrix} 1 \\ -i \end{pmatrix} e^{(1+i)t} = e^t \begin{pmatrix} 1 \\ -i \end{pmatrix}(\cos t + i \sin t)$

$= e^t \left[\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} + i\begin{pmatrix} 0 \\ -1 \end{pmatrix}\right)(\cos t + i \sin t)\right]$

$= e^t \left[\begin{pmatrix} \cos t \\ \sin t \end{pmatrix} + i\begin{pmatrix} \sin t \\ -\cos t \end{pmatrix}\right]$

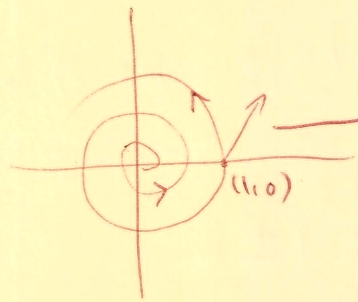
$\Rightarrow \tilde{x}(t) = c_1 \begin{pmatrix} e^t \cos t \\ e^t \sin t \end{pmatrix} + c_2 \begin{pmatrix} e^t \sin t \\ -e^t \cos t \end{pmatrix}$

$\begin{pmatrix} x \\ y \end{pmatrix}$ ″

(b) (15 points) Sketch the phase portrait. Summarize the main features and explain why it looks that way. Specify what kind of point the origin is (e.g. spiral sink, nodal source, etc.)

spiral b/c complex eigenvalues

$\mathrm{Re}(1 \pm i) = 1 > 0$



$\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

thus the direction is counter-clockwise

unstable because $\mathrm{Re}(1 \pm i) = 1 > 0$

(c) (10 points) Find the solution with initial condition $x(0) = 1, y(0) = -1$.

$x(0) = 1 \Rightarrow 1 = c_1 \cdot 1 + c_2 \cdot 0$

$y(0) = -1 \Rightarrow -1 = -c_2$

$\Rightarrow x(t) = e^t \begin{pmatrix} \cos t \\ \sin t \end{pmatrix} + e^t \begin{pmatrix} \sin t \\ -\cos t \end{pmatrix}$