1. (20 points). Solve the separable differential equation

$$\frac{dy}{dx} = \frac{e^x}{y^2}.$$

$$y^2 \frac{dy}{dx} = e^x$$

$$\int y^2\, dy = \int e^x\, dx$$

$$\tfrac{1}{3} y^3 = e^x + C$$

$$y = (3e^x + C)^{1/3}$$

2. (20 points). Solve the separable differential equation

$$xy' - 2y = y^2.$$

$$y' = \frac{y^2 + 2y}{x}$$

$$\int \frac{A}{y+2} + \frac{B}{y}\, dy = \int \frac{1}{y^2+2y}\, dy = \int \frac{1}{x}\, dx$$

$$\begin{cases} A = -1/2 \\ B = 1/2 \end{cases}$$

$$-\tfrac{1}{2}\ln|y+2| + \tfrac{1}{2}\ln|y| = \ln|x| + C$$

$$\ln\left|\left(\frac{y}{y+2}\right)^{1/2}\right| = \ln|x| + C$$

$$\left(\frac{y}{y+2}\right)^{1/2} = Cx^2$$

$$\frac{y}{y+2} = Cx^2$$

$$y = Cx^2(y+2)$$

$$y - Cx^2 y = 2Cx^2$$

$$y = \frac{2Cx^2}{1 - Cx^2}$$

3. (20 points). Solve the linear equation

$$\frac{dy}{dx} + \frac{2x}{1+x^2}\, y = e^x.$$

$$\left(IF = e^{\int \frac{2x}{1+x^2}\,dx} = e^{\ln|1+x^2|} = 1+x^2 \right)$$

$$\int D_x\left[(1+x^2)y\right]dx = \int (1+x^2)e^x\,dx$$

$$\left(\begin{array}{ll} \text{Int. by parts} & \\ u = x^2 & du = 2x\,dx \\ dv = e^x dx & v = e^x \end{array} \right)$$

$$(1+x^2)y = e^x + x^2e^x - \int 2xe^x\,dx$$

$$\left(\begin{array}{ll} \text{Int. by parts} & \\ u = 2x & du = 2\,dx \\ dv = e^x dx & v = e^x \end{array} \right)$$

$$= e^x + x^2e^x - 2xe^x + 2e^x + C$$

$$y = \frac{e^x}{1+x^2}(x^2 - 2x + 3) + \frac{C}{1+x^2}$$

4. (15+25 points). (a). Solve the homogeneous second-order linear equation

$$y'' + 4y' + 4y = 0.$$

(b). Solve the initial value problem

$$y'' + 4y' + 4y = \sin 2x, \quad y(0) = -\frac{1}{8}, \quad y'(0) = 1.$$

(a) Auxillary Equation: $r^2 + 4r + 4 = 0$

$(r+2)^2 = 0$

$r_1 = r_2 = r = -2$

$$\boxed{y = c_1 e^{-2x} + c_2 x e^{-2x}}$$

(b) $y_p = A \sin 2x + B \cos 2x$

$y_p' = -2B \sin 2x + 2A \cos 2x$

$y_p'' = -4A \sin 2x - 4B \cos 2x$

$$\boxed{y = xe^{-2x} - \tfrac{1}{8} \cos 2x}$$

$\cos 2x \rightsquigarrow -4B + 8A + 4B = 0 \Rightarrow A = 0$

$\sin 2x \rightsquigarrow -4A - 8B + 4A = 1 \Rightarrow B = -\tfrac{1}{8}$

$y = y_c + y_p = c_1 e^{-2x} + c_2 x e^{-2x} - \tfrac{1}{8} \cos 2x$

$y' = (-2c_1 + c_2) e^{-2x} - 2c_2 x e^{-2x} + \tfrac{1}{4} \sin 2x$

$y(0) = -\tfrac{1}{8} \Rightarrow -\tfrac{1}{8} = c_1 - \tfrac{1}{8} \Rightarrow c_1 = 0$

$y'(0) = 1 \Rightarrow 1 = -2c_1 + c_2 \Rightarrow c_2 = 1$