(11) This problem is #16 in previous review. It snuck into this version by accident.

(12) a) Im(A) has as a basis the "leading columns":

$$\begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix},$$

$\dim(\mathrm{im}(A)) = 2$

The matrix A represents a transformation $\mathbb{R}^5 \to \mathbb{R}^3$

So $\mathrm{Im}(A)$ is a 2-dimensional subspace of $\mathbb{R}^3$

b) The kernel of A consists of the relations given by the free variables (of which there are three):

Basis is $\begin{pmatrix} -2 \\ +1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -1 \\ 0 \\ +1 \end{pmatrix}$

$\ker(A)$ has dim 3 and is a subspace of $\mathbb{R}^5$

(13) We need only find another vector which is <u>not</u> a linear combination of the 1^{st} two. There are many ways;

<u>method 1</u>: Calculate $\vec{w} = \vec{v}_1 \times \vec{v}_2 = \begin{pmatrix} 8 \\ -3 \\ -6 \end{pmatrix}$
this perpendicular to both, so is independent of them

<u>method 2</u>: Guess a 3^{rd} column to the matrix $\begin{pmatrix} 1 & 1 & \\ 2 & -4 & \\ -1 & 2 & \end{pmatrix}$ so that determinant is non-zero.
eg. $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$; then expanding along 3^{rd} col. gives determinant 8.

probably others...

(14)

$$B_1 : \begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}1\\2\end{pmatrix}$$

$$B_2 : \begin{pmatrix}1\\2\end{pmatrix}, \begin{pmatrix}1\\3\end{pmatrix}$$

Note: in this question, there shouldn't be { } on the bases.

$$\left[\begin{pmatrix}1\\1\end{pmatrix}\right]_{B_2} \rightarrow \quad \begin{pmatrix}1\\1\end{pmatrix} = \underline{2}\begin{pmatrix}1\\2\end{pmatrix} - \underline{1}\begin{pmatrix}1\\3\end{pmatrix} \quad \text{by inspection}$$

$$\text{So} \quad \left[\begin{pmatrix}1\\1\end{pmatrix}\right]_{B_2} = \begin{pmatrix}2\\-1\end{pmatrix}$$

$$\left[\begin{pmatrix}1\\2\end{pmatrix}\right]_{B_2} \rightarrow \quad \begin{pmatrix}1\\2\end{pmatrix} = \underline{1}\begin{pmatrix}1\\2\end{pmatrix} + \underline{0}\begin{pmatrix}1\\3\end{pmatrix} \quad \text{by inspection}$$

$$\text{So} \quad \left[\begin{pmatrix}1\\2\end{pmatrix}\right]_{B_2} = \begin{pmatrix}1\\0\end{pmatrix}$$

$$\text{Therefore} \quad S_{B_1 \rightarrow B_2} = \left[\left[\begin{pmatrix}1\\1\end{pmatrix}\right]_{B_2} \left[\begin{pmatrix}1\\2\end{pmatrix}\right]_{B_2} \right]$$

$$= \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}$$

(15)

$$B_1 : \vec{v}_1, \vec{v}_2$$

$$B_2 : \vec{v}_2, 2\vec{v}_1$$

$$[\vec{v}_1]_{B_2} \Longleftrightarrow \quad \vec{v}_1 = \underline{0}(\vec{v}_2) + \underline{\tfrac{1}{2}}(2\vec{v}_1) \quad \text{by inspection}$$

$$[\vec{v}_1]_{B_2} = \begin{pmatrix}0\\1/2\end{pmatrix}$$

$$[\vec{v}_2]_{B_2} \longrightarrow \quad \vec{v}_2 = 1\,(\vec{v}_2) + 0\,(2\vec{v}_1) \quad \text{by inspection}$$

$$[\vec{v}_2]_{B_2} = \begin{pmatrix}1\\0\end{pmatrix}$$

$$S_{B_1 \rightarrow B_2} = \begin{pmatrix} 0 & 1 \\ 1/2 & 0 \end{pmatrix}$$

(16) This is a little beyond what we've covered in the course. Consider it a "challenge problem" (and challenge problems don't end up on exams).

$$[T]_{B \text{ and } B'} = \left[\left[T\begin{pmatrix}2\\1\end{pmatrix}\right]_{B'}, \left[T\begin{pmatrix}1\\2\end{pmatrix}\right]_{B'} \right]$$

$$= \left[\left[\begin{pmatrix}1\\2\\1\end{pmatrix}\right]_{B'}, \left[\begin{pmatrix}2\\1\\-1\end{pmatrix}\right]_{B'} \right]$$

$$= \begin{bmatrix} 1 & -1 \\ 1 & 2 \\ -1 & 1 \end{bmatrix}$$

since $\begin{pmatrix}2\\1\\-1\end{pmatrix} = -1\begin{pmatrix}1\\1\\1\end{pmatrix} + 2\begin{pmatrix}1\\1\\0\end{pmatrix} + 1\begin{pmatrix}1\\0\\0\end{pmatrix}$

since $\begin{pmatrix}1\\2\\1\end{pmatrix} = 1\begin{pmatrix}1\\1\\1\end{pmatrix} + 1\begin{pmatrix}1\\1\\0\end{pmatrix} - 1\begin{pmatrix}1\\0\\0\end{pmatrix}$

(17) Ooops. This question doesn't actually make sense in the context of what we've done. In case you're curious,

It's claiming $M = [S]_{std}\, S_{B \to std}$.

Notice the first $S_{std \to B}$ is missing. This means M takes <u>in</u> vectors with respect to basis B but gives them <u>out</u> in the standard basis. Let $S_{B \to std} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, and solve $\begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} \Rightarrow \begin{pmatrix} 8/3 & 5/3 \\ -1/3 & -1/3 \end{pmatrix}$

So basis is $\begin{pmatrix} 8/3 \\ -1/3 \end{pmatrix}\begin{pmatrix} 5/3 \\ -1/3 \end{pmatrix}$.

(18) a) To check, put them in a matrix

$$A = \begin{pmatrix} 1 & 0 & 3 \\ 3 & -1 & 3 \\ -5 & 3 & 3 \end{pmatrix}$$

Check if A is invertible. For example, by row-reducing, or

$$\begin{aligned} \det(A) &= 1\left(-3-9\right) + 3\left(9-5\right) \\ &= 1\left(-12\right) + 3(4) \\ &= -12 + 12 \\ &= 0 \end{aligned}$$

So the vectors do <u>not</u> form a basis. In fact,

$$\begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix} = 3\begin{pmatrix} 1 \\ 3 \\ -5 \end{pmatrix} + 6\begin{pmatrix} 0 \\ -1 \\ 3 \end{pmatrix}$$

(19) Again, there are three, so they span $\Longleftrightarrow$ they are linearly independent.

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ -1 & 1 & 4 \end{pmatrix} \qquad \begin{aligned} \det(A) &= 1\left(4-2\right) - 1\left(2-1\right) \\ &= 2 - 1 = 1 \end{aligned}$$

So they <u>do</u> span $\mathbb{R}^3$.

(20) $A\vec{v}_i = \vec{e}_i$ for $i=1,2,3$

$\iff$ $\vec{v}_i = A^{-1}\vec{e}_i$ for $i=1,2,3$ (For A^{-1} to exist, the vectors v_i must be independent — look at zeroes to see it is so)

So

A^{-1} has columns $\begin{pmatrix}2\\0\\0\end{pmatrix}$, $\begin{pmatrix}1\\3\\0\end{pmatrix}$ and $\begin{pmatrix}0\\1\\1\end{pmatrix}$

i.e. $A^{-1} = \begin{pmatrix}2&1&0\\0&3&1\\0&0&1\end{pmatrix}$. So find A by row-reduction algorithm.

$$A = \frac{1}{6}\begin{pmatrix}3&-1&1\\0&2&-2\\0&0&6\end{pmatrix}$$

<u>Check</u>: $\frac{1}{6}\begin{pmatrix}3&-1&1\\0&2&-2\\0&0&6\end{pmatrix}\begin{pmatrix}2\\0\\0\end{pmatrix} = \frac{1}{6}\begin{pmatrix}6\\0\\0\end{pmatrix} = \begin{pmatrix}1\\0\\0\end{pmatrix}$

$(A\vec{v}_i = \vec{e}_i)$ $\frac{1}{6}\begin{pmatrix}3&-1&1\\0&2&-2\\0&0&6\end{pmatrix}\begin{pmatrix}1\\3\\0\end{pmatrix} = \frac{1}{6}\begin{pmatrix}0\\6\\0\end{pmatrix} = \begin{pmatrix}0\\1\\0\end{pmatrix}$

$\frac{1}{6}\begin{pmatrix}3&-1&1\\0&2&-2\\0&0&6\end{pmatrix}\begin{pmatrix}0\\1\\1\end{pmatrix} = \frac{1}{6}\begin{pmatrix}0\\0\\6\end{pmatrix} = \begin{pmatrix}0\\0\\1\end{pmatrix}$

<u>Note</u>: A is $S_{std \to B}$

A^{-1} is $S_{B \to std} = \begin{pmatrix}[\vec{v}_1]_{std} & [\vec{v}_2]_{std} & [\vec{v}_3]_{std}\end{pmatrix}$
$= \begin{pmatrix}2&1&0\\0&3&1\\0&0&1\end{pmatrix}$

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(21) $T(a+bx+cx^2+dx^3)$
$= x^2(2c+6dx) + a+bx+cx^2+dx^3$
$= a + bx + (3c)x^2 + (7d)x^3$

$\mathcal{B}: 1, x, x^2, x^3$

$$[T]_{\mathcal{B}} = \begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&3&0\\0&0&0&7\end{bmatrix}$$

(22) $$\begin{bmatrix} 0 & 1 & 2 & -1 \\ 2 & 5 & -7 & 3 \\ 0 & 3 & 6 & 2 \\ -2 & -5 & 4 & -2 \end{bmatrix} \xrightarrow[\text{reduce}]{\text{row}} I_4$$

So yes, they are linearly independent.

(23) $$A = \begin{bmatrix} 1 & 3 & 1 & 3 \\ 2 & 1 & -3 & 1 \\ 0 & 1 & 1 & -2 \\ 0 & 2 & 2 & 1 \end{bmatrix} \xrightarrow[\text{reduce}]{\text{row}} \begin{array}{c} \begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\ \begin{array}{cccc} L & L & F & L \end{array} \end{array}$$

a) $$\ker(A) = \left\{ \begin{pmatrix} 2t \\ -t \\ t \\ 0 \end{pmatrix} \right\} \quad \text{Basis: } \begin{pmatrix} 2 \\ -1 \\ 1 \\ 0 \end{pmatrix}.$$

b) No, since there are free variables. The elements of the kernel give the linear relations. So we have

$$2\begin{pmatrix} 1 \\ 2 \\ 0 \\ 0 \end{pmatrix} - 1\begin{pmatrix} 3 \\ 1 \\ 1 \\ 2 \end{pmatrix} + 1\begin{pmatrix} 1 \\ -3 \\ 1 \\ 2 \end{pmatrix} + 0\begin{pmatrix} 3 \\ 1 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

(24) $$A = \begin{bmatrix} 1 & 3 & -1 & -2 \\ 2 & 4 & -1 & -1 \\ -3 & -9 & h & 6 \end{bmatrix}$$

$T(\vec{x}) = A\vec{x}$

$T: \mathbb{R}^4 \to \mathbb{R}^3$

Row Reduce:

$$\rightsquigarrow \begin{bmatrix} 1 & 3 & -1 & -2 \\ 0 & -2 & 1 & 3 \\ 0 & 0 & h-3 & 0 \end{bmatrix}$$

$$\rightsquigarrow \begin{bmatrix} 1 & 3 & -1 & -2 \\ 0 & 1 & -1/2 & -3/2 \\ 0 & 0 & h-3 & 0 \end{bmatrix}$$

$$\rightsquigarrow \begin{bmatrix} 1 & 0 & 1/2 & 5/2 \\ 0 & 1 & -1/2 & -3/2 \\ 0 & 0 & h-3 & 0 \end{bmatrix}$$

At this point we can see that we obtain a 3^{rd} leading one if and only if $h \neq 3$.

a) So $\operatorname{rank}(A) = 3 \iff h \neq 3$ i.e. T is onto (surjective) if and only if $h \neq 3$

b) Now by rank-nullity, $\operatorname{nullity}(A) = \dim(\mathbb{R}^4) - \operatorname{rank}(A) = 4 - \operatorname{rank}(A)$. But $\operatorname{rank}(A) \leq 3$ always, so $\operatorname{nullity}(A) \geq 1$. So T is <u>never</u> 1-1 (injective).

(25) $A = \begin{bmatrix} -2 & 1 & 0 \\ 3 & 3 & 1 \\ 1 & -1 & 0 \end{bmatrix}$

(26) a) $T(\vec{e}_1) = \vec{e}_1$

$T(\vec{e}_2) = \vec{e}_1 + \vec{e}_2$

So the 1^{st} column of $\underset{\text{matrix}}{T\text{'s}}$ is $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

and 2^{nd} column of $\underset{\text{matrix}}{T\text{'s}}$ is $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$.

In order for T to be invertible, it's matrix must be invertible. Suppose the 3^{rd} column is $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

Then $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

and A row-reduces to $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (in one step!)

So A has <u>full rank</u>, and so is invertible.

(27) a) $S = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} : x_1 = 2x_3 - x_2 + 1 \text{ and } x_1 + 3x_2 = 0 \right\}$

Example vectors in S: $\vec{u}_1 = \begin{pmatrix} -3 \\ 1 \\ -3/2 \end{pmatrix}$, $\vec{u}_2 = \begin{pmatrix} 0 \\ 0 \\ -1/2 \end{pmatrix}$

Now $\vec{u}_1 + \vec{u}_2 = \begin{pmatrix} -3 \\ 1 \\ 3/4 \end{pmatrix}$ and $2\vec{u}_2 = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$

are <u>not</u> in S.

So this is <u>not</u> a vector subspace.

(Intuitively, the problem is the "+1")

b) $T = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : ad - bc = 0 \right\}$

example vectors in T: $\vec{u}_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $\vec{u}_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

But $\vec{u}_1 + \vec{u}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is <u>not</u> in T

So T is <u>not</u> a vector subspace.

(28) a) $F\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}$

$F\begin{pmatrix} 0 \\ 0 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \\ 1 \end{pmatrix}$

<u>Not</u> a linear transformation since

$F\begin{pmatrix} 0 \\ 0 \\ 0 \\ 2 \end{pmatrix} \neq 2F\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$

b) $F(A) = A^T$: <u>is</u> a linear transformation since

$$(A+B)^T = A^T + B^T$$
$$(kA)^T = kA^T$$

$F(A) = A$ <u>is</u> a linear transformation

So $F(A) = A^T - 5A$ is a linear transformation because it is a linear combination of the previous two.

c) This is a linear transformation because

$$\begin{aligned} F(p(t)+q(t)) &= 2(p(t)+q(t))' + (p(0)+q(0)) \\ &= 2p'(t) + 2q'(t) + p(0) + q(0) \\ &= F(p(t)) + F(q(t)) \end{aligned}$$

and
$$\begin{aligned} F(kp(t)) &= 2(kp(t))' + kp(0) \\ &= k(2p'(t) + p(0)) \\ &= kF(p(t)) \end{aligned}$$

(29) a) $A \xrightarrow[\text{reduce}]{\text{row}} \begin{bmatrix} 1 & 0 & -1 & 5/2 \\ 0 & 1 & 2 & -3/4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$$\begin{array}{cccc} L & L & F & F \end{array}$$

$\operatorname{Im}(A)$ has basis $\begin{pmatrix} 1 \\ 5 \\ 9 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 6 \\ 10 \\ 14 \end{pmatrix}$

$$\operatorname{Ker}(A) = \left\{ \begin{pmatrix} s - \frac{5}{2}t \\ -2s + \frac{3}{4}t \\ s \\ t \end{pmatrix} \right\} = \operatorname{Span}\left\{ \begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -5/2 \\ 3/4 \\ 0 \\ 1 \end{pmatrix} \right\}$$

and has basis $\begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -5/2 \\ 3/4 \\ 0 \\ 1 \end{pmatrix}$.

(30) We put these into a matrix

$$\begin{bmatrix} 1 & 0 & -3 & -1 & 2 \\ 2 & 1 & 2 & -2 & 4 \\ 0 & -1 & -8 & 0 & 2 \\ 1 & 0 & -3 & -1 & 2 \end{bmatrix}$$

$\underbrace{\hphantom{1 \quad 0 \quad -3 \quad -1}}_{\text{spanning set for } H}$ polynomial in question

Then row-reduce to obtain: $\begin{bmatrix} 1 & 0 & -3 & -1 & 0 \\ 0 & 1 & 8 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

$$\begin{array}{ccccc} L & L & F & F & L \end{array}$$

a) Since the last column is leading, the polynomial $2 + 4t + 2t^2 + 2t^3$ does not belong to H (since it is not a linear combination of earlier ones).

b) A basis for H is $1 + 2t + t^3$, $t - t^2$, the only leading columns of the first four.