(31)

a) $A = \begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix}$ $\quad \det(A) = 1\,(28 - 12) + 2(-12-4)$
$= 16 - 32 = -16$

Since $\det(A) \neq 0$, A is invertible and B is a basis (could also row-reduce to show this).

$S_{B \to std} = A$ which we already have!

b) $S_{std \to B} = A^{-1}$ To find this we can row-reduce

$$= \frac{1}{16}\begin{bmatrix} 12 & -5 & -6 \\ 8 & -6 & -4 \\ -16 & 16 & 16 \end{bmatrix}$$

(32)

a) $\begin{bmatrix} 1 & 0 & 1 \\ 3 & 1 & 0 \\ 4 & 3 & -2 \end{bmatrix} \xrightarrow[\text{reduce}]{\text{row}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ independent

b) $\begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \\ 3 & 6 & 9 \end{bmatrix} \xrightarrow[\text{reduce}]{\text{row}} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$ dependent

$\vec{v}_3 = -\vec{v}_1 + 2\vec{v}_2$

(33)

a) $A = \begin{bmatrix} 1 & -1 & 1 & -2 & 1 \\ 1 & -1 & 1 & -3 & 2 \\ 2 & -1 & 2 & -1 & 1 \\ 3 & -2 & 3 & -4 & 3 \end{bmatrix} \xrightarrow[\text{reduce}]{\text{row}} \begin{array}{c} \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\ \begin{matrix} L & L & F & L & F \end{matrix} \end{array}$

b) $\text{kernel}(A) = \left\{ \begin{pmatrix} -s-t \\ -2t \\ s \\ t \\ t \end{pmatrix} \right\}$ Basis: $\begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ -2 \\ 0 \\ 1 \\ 1 \end{pmatrix}$.

c) $\text{rank}(A) = 3$; $\text{nullity} = \dim(\ker(A)) = 2$

d) For $T: V \to W$, $\dim(V) = \text{rank}(T) + \text{nullity}(T)$

Here, $T: \mathbb{R}^5 \to \mathbb{R}^4$ $\quad 5 = 3 + 2$ ✓

(34) $V = \mathbb{R}^{2\times 2}$

a) $S_2 = \{A \in V;\ \det(A) = 0\}$

FALSE. In particular, $\begin{pmatrix}1&0\\0&0\end{pmatrix} \in S_2$ and $\begin{pmatrix}0&0\\0&1\end{pmatrix} \in S_2$

since they have determinant 0.

However, $\begin{pmatrix}1&0\\0&0\end{pmatrix} + \begin{pmatrix}0&0\\0&1\end{pmatrix} = \begin{pmatrix}1&0\\0&1\end{pmatrix}$ has determinant 1.

Therefore, $\begin{pmatrix}1&0\\0&1\end{pmatrix}$ is not in S_2.

So S_2 is not closed under addition, and hence is <u>not a subspace of V</u>.

b) Let $L: V \to V$ be defined by $L(A) = A^T$.

TRUE. Then $L\begin{pmatrix}a&b\\c&d\end{pmatrix} = \begin{pmatrix}a&c\\b&d\end{pmatrix}$.

Let $\begin{pmatrix}a&b\\c&d\end{pmatrix}$ and $\begin{pmatrix}e&f\\g&h\end{pmatrix}$ be elements of V, $k \in \mathbb{R}$.

Then

$$\begin{aligned}
L\left(\begin{pmatrix}a&b\\c&d\end{pmatrix} + \begin{pmatrix}e&f\\g&h\end{pmatrix}\right) &= L\begin{pmatrix}a+e&b+f\\c+g&d+h\end{pmatrix} \\
&= \begin{pmatrix}a+e&c+g\\b+f&d+h\end{pmatrix} \\
&= \begin{pmatrix}a&c\\b&d\end{pmatrix} + \begin{pmatrix}e&g\\f&h\end{pmatrix} \\
&= L\begin{pmatrix}a&b\\c&d\end{pmatrix} + L\begin{pmatrix}e&f\\g&h\end{pmatrix}
\end{aligned}$$

and

$$\begin{aligned}
L\left(k\begin{pmatrix}a&b\\c&d\end{pmatrix}\right) &= L\begin{pmatrix}ka&kb\\kc&kd\end{pmatrix} \\
&= \begin{pmatrix}ka&kc\\kb&kd\end{pmatrix} = k\begin{pmatrix}a&c\\b&d\end{pmatrix} \\
&= k\,L\begin{pmatrix}a&b\\c&d\end{pmatrix}
\end{aligned}$$

Therefore L is a linear transformation.

(35) a) $S = \{ \text{invertible } 2\times 2 \text{ matrices} \} \subset \mathbb{R}^{2\times 2}$

NOT A SUBSPACE

$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in S$ and $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in S$

But $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \notin S$

(not closed under addition)

b) $S = \left\{ \begin{pmatrix} a & 0 \\ b & c \end{pmatrix} \right\} \subset \mathbb{R}^{2\times 2}$

SUBSPACE

If $\begin{pmatrix} a & 0 \\ b & c \end{pmatrix}, \begin{pmatrix} d & 0 \\ e & f \end{pmatrix} \in S$

then $\begin{pmatrix} a & 0 \\ b & c \end{pmatrix} + \begin{pmatrix} d & 0 \\ e & f \end{pmatrix} = \begin{pmatrix} a+d & 0 \\ b+e & c+f \end{pmatrix} \in S$

and $k\begin{pmatrix} a & 0 \\ b & c \end{pmatrix} = \begin{pmatrix} ka & 0 \\ kb & kc \end{pmatrix} \in S$

so S is closed under addition and scalar multiplication.

In addition, $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \in S$.

So S is a subspace.

c) $S = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : ad - bc = 1 \right\} \subset \mathbb{R}^{2\times 2}$

NOT A SUBSPACE

$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \notin S$ since $\det \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \neq 1$

So S doesn't contain the zero element.

(36) IGNORE THIS QUESTION (we haven't covered this yet)

(37) First, $\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$ are in S since $1 + (-1) = 0$.

Second, they are independent since they are not multiples of each other.

Since S is a plane, a basis must have 2 elements (dim. of plane is 2).

Since these 2 are independent, they form a basis.

(38) True by rank-nullity theorem.

(39)

a) $\text{rref}\begin{bmatrix} 3 & -2 & -1 \\ 1 & 1 & 3 \\ 2 & 3 & 8 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$ free variable.

So these are linearly <u>dependent</u>.

b) <u>Independent</u> (fact from calculus.)

c) <u>Dependent</u> since $\cos^2 x + \sin^2 x = 1$
i.e. $g_2 + g_1 = g_3$

d) <u>Independent</u> by considering the many zeroes: moving from right to left, each new vector cannot be formed from previous ones.

e) <u>Dependent</u> because these are 4 vectors in $\mathbb{R}^3$. Since $\mathbb{R}^3$ has dimension 3, it cannot contain 4 independent vectors.

(40) a) TRUE.

If S, T are subspaces, then $S \cap T$ is a subspace.

Proof: If $\vec{v}, \vec{w} \in S \cap T$, $k \in \mathbb{R}$, then
$\vec{v}, \vec{w} \in S \Rightarrow \vec{v}+\vec{w} \in S$, $k\vec{v} \in S$ because S is a subspace.
Also, $\vec{v}, \vec{w} \in T \Rightarrow \vec{v}+\vec{w} \in T$, $k\vec{v} \in T$ because T is a subspace.
So $\vec{v}+\vec{w}, k\vec{v} \in S \cap T$.
Also, $\vec{0} \in S$ and $\vec{0} \in T$ So $\vec{0} \in S \cap T$.
Therefore $S \cap T$ is closed under addition and scalar multiplication and contains zero.

b) FALSE.

Counterexample: $S = \text{span}\left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right\}$, $T = \text{span}\left\{\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right\}$.
Then $\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \in S \cup T$ but $\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \notin S \cup T$.
So $S \cup T$ is not a subspace.

(41) a) The vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ are <u>dependent</u> if there exist $a_1, a_2, \ldots, a_n \in \mathbb{R}$, not all zero, such that

$$a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_n\vec{v}_n = \vec{0}.$$

b) The vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ form a basis for the linear space V if $V = \operatorname{span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ and $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ are linearly independent.

c) Let V and W be linear spaces. A function $T: V \to W$ is a <u>linear transformation</u> if for any $\vec{v}_1, \vec{v}_2 \in V$ and $k \in \mathbb{R}$,

$$T(\vec{v}_1 + \vec{v}_2) = T(\vec{v}_1) + T(\vec{v}_2)$$

and $T(k\vec{v}_1) = kT(\vec{v}_1)$

(42) a) $\begin{bmatrix} -4 & 1 & -3 \\ 1 & 2 & 3 \\ 3 & 5 & 8 \end{bmatrix} \xrightarrow[\text{reduce}]{\text{row}} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

$\Rightarrow$ they are <u>not</u> linearly independent

b) $\begin{bmatrix} 1 & 1 & 1 \\ 2 & 4 & 1 \\ 1 & -1 & 2 \end{bmatrix} \xrightarrow[\text{reduce}]{\text{row}} \begin{bmatrix} 1 & 0 & 3/2 \\ 0 & 1 & -1/2 \\ 0 & 0 & 0 \end{bmatrix}$

$\Rightarrow$ they are <u>not</u> linearly independent

c) $\begin{bmatrix} 1 & 2 & 1 & 0 \\ 2 & 1 & -1 & 1 \\ -1 & 1 & 2 & 2 \\ 0 & -2 & 0 & 0 \end{bmatrix} \xrightarrow[\text{reduce}]{\text{row}} I_4$

$\Rightarrow$ they <u>are</u> linearly independent

(43) $L : \mathbb{R}^3 \to \mathbb{R}^4$

$$L(\vec{x}) = A\vec{x} \qquad A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 0 & -1 \\ -1 & 3 & 4 \\ 1 & 3 & 2 \end{bmatrix}$$

$$\text{Row reduce} \longrightarrow \begin{array}{c} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ \begin{matrix} L & L & F \end{matrix} \end{array}$$

a) $\ker(A) = \left\{ \begin{pmatrix} t \\ -t \\ t \end{pmatrix} \right\}$ Basis: $\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$

Basis for image: $\begin{pmatrix} 1 \\ 1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 3 \\ 3 \end{pmatrix}$ $(\text{rank}(A) = 2)$

b) $L : \mathbb{R}^3 \to \mathbb{R}^4$

$$\dim(\mathbb{R}^3) = \text{rank}(A) + \text{nullity}(A)$$
$$3 = 2 + 1 \quad \checkmark$$

(44) a) $T(a+bx+cx^2+dx^3) = b+2cx+3dx^2+a+bx+cx^2+dx^3$
$= (a+b) + (b+2c)x + (c+3d)x^2 + dx^3$

$\mathcal{B}: 1, x, x^2, x^3$

$$[T]_{\mathcal{B}} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow[\text{reduce}]{\text{row}} \begin{bmatrix} 1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 1 \end{bmatrix}$$

So T does have an inverse since it is full rank.
It is $[T^{-1}]_{\mathcal{B}} = \begin{bmatrix} 1 & -1 & 2 & -6 \\ 0 & 1 & -2 & 6 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

So $T^{-1}(a+bx+cx^2+dx^3) = a-b+2c-6d$
$+ (b-2c+6d)x + (c-3d)x^2 + x^3$

(45) If A is 64×17 of rank 11, it has 11 leading variables and $17 - 11 = 6$ free variables. So $\ker(A)$ has a basis of 6 elements. So 6 linearly independent vectors satisfy $A\vec{x} = \vec{0}$.

This part is beyond what we've done in the course.

It is a fact that A^T has the same rank as A.

So in this case A^T is 17×64 and there are $64 - 11 = 53$ free variables. So the answer is 53.

(46) $\det(A) = 8$

$\det(B) = 0$ (2^{nd} column is multiple of 1^{st})

So since similar matrices have equal determinant, they are not.

If you prefer not to use determinants, note
$\text{rank}(A) = 3$, $\text{rank}(B) = 2$,
i.e. A invertible, B not.

So they cannot be similar.

(47) $T(p) = 3p - 2p'$ $\quad \mathcal{B}: 1, t, t^2$

$$\begin{aligned} T(a + bt + ct^2) &= 3a + 3bt + 3ct^2 - 2b - 4ct \\ &= (3a - 2b) + (3b - 4c)t + 3ct^2 \end{aligned}$$

So $[T]_{\mathcal{B}} = \begin{bmatrix} 3 & -2 & 0 \\ 0 & 3 & -4 \\ 0 & 0 & 3 \end{bmatrix}$.