Step and delta functions; step and delta responses

1. Suppose $q(t) = 2u(t-1) + \delta(t-2) - 2u(t-3)$. Sketch a graph of this generalized function. Formulate at least one scenario which might result in each of the equations

   $x' + kx = q(t)$ (your choice of k, it might be negative);

   $2x'' + 4x' + 4x = q(t)$.

   The graph will have a spike of 1at $t = 2$ because of the delta. Other than that, for $t < 1$ both step functions are 0, so $q(t) = 0$. For $1 < t < 2$ and $2 < t < 3$ the first step function is 1, while the second is 0, so $q(t) = 2$. Finally, for $t > 3$ both step functions are 1 and they cancel each other, so $q(t) = 0$.

2. (a) Graph the functions

   $$f(t) = 3\,(u(t-1) - u(t-2))\,t$$

   and

   $$g(t) = \begin{cases} 0 & t < 0 \\ \lfloor t \rfloor & t > 0 \end{cases}$$

   where $\lfloor t \rfloor$ denotes the greatest integer less than or equal to t.

   left for you

   (b) Express $f(t)$ as an alternative ($f(t) = \cdots$ for $t < \cdots$, etc). Express $g(t)$ as a single formula using the step function $u(t)$.

   $$f(t) = \begin{cases} 0 & t < 1 \\ 3t & 1 < t < 2 \\ 0 & t > 2 \end{cases}$$

   $$g(t) = u(t-1) + u(t-2) + \cdots = \sum_{k=1}^{\infty} u(t-k)$$

(c) Using the graphs of $f(t)$ and $g(t)$, graph the generalized derivatives of these two functions. Use labeled harpoons to denote the delta functions that occur.

Once again, I leave the graphing to you.

(d) Finally, differentiate formally the expression for $f(t)$ that was given and the expression for $g(t)$ you found in part (b). Graph the resulting functions.

Applying the product rule,

$$\begin{aligned} f'(t) &= 3\left(u'(t-1) - u'(t-2)\right) t + 3\left(u(t-1) - u(t-2)\right) \cdot 1 \\ &= 3\left(\delta(t-1) - \delta(t-2)\right) t + 3\left(u(t-1) - u(t-2)\right) \end{aligned}$$

Now we use a key fact about delta,

$$\boxed{f(t)\delta(t-a) = f(a)\delta(t-a)}$$

and the derivative becomes

$$f'(t) = 3\delta(t-1) - 3\delta(t-2) + 3u(t-1) - 3u(t-2)$$

3. Find the unit step function and unit impulse responses to the operator $D^2 + 2D + 2I$, and graph them. Why is one the derivative of the other? How do these results change if one uses instead the operator $2D^2 + 4D + 4I$?

Step response

We need to solve the equation

$$x'' + 2x' + 2x = u(t).$$

with *rest initial conditions*. That means that x'' must match the singularity of $u(t)$ and have a jump of 1 at $t = 0$. Which in turn implies that both x' and x

are continuous functions and therefore take the value 0 at $t = 0$. So we need to solve the equation

$$x'' + 2x' + 2x = 1 \quad \text{for } t > 0, \qquad x(0+) = 0, x'(0+) = 0.$$

Now ERF says that a particular solution is $x_p(t) = \frac{1}{p(0)} = \frac{1}{2}$.
Since the roots of p are $-1 \pm i$, the homogeneous part is $x_h(t) = c_1 e^{-t} \cos t + c_2 e^{-t} \sin t$.
Thus, the general solution is

$$x(t) = \frac{1}{2} + c_1 e^{-t} \cos t + c_2 e^{-t} \sin t.$$

Plugging in the initial conditions we get $c_1 = c_2 - \frac{1}{2}$. So the unit response function is

$$v(t) = \begin{cases} \frac{1}{2} - \frac{1}{2}e^{-t}\cos t - \frac{1}{2}e^{-t}\sin t & t > 0, \\ 0 & t < 0. \end{cases}$$

Unit impulse response

We need to solve the equation

$$x'' + 2x' + 2x = \delta(t).$$

with *rest initial conditions*. That means that x'' must match the singularity of $\delta(t)$. This implies that x', being the antiderivative of x'', has a jump of 1 at $t = 0$. Which in turn implies that x is continuous and therefore takes the value 0 at $t = 0$. So we need to solve the equation

$$x'' + 2x' + 2x = 0 \quad \text{for } t > 0, \qquad x(0+) = 0, x'(0+) = 1.$$

This is now a homogeneous equation with general solution $x(t) = c_1 e^{-t} \cos t + c_2 e^{-t} \sin t$. The initial conditions give $c_1 = 0$ and $c_2 = 1$. So the unit impulse response is

$$w(t) = \begin{cases} e^{-t}\sin t & t > 0, \\ 0 & t < 0. \end{cases}$$

Since $\delta(t) = u'(t)$, it follows that the same relationship must be satisfied by the system responses to these two inputs, so $w(t) = v'(t)$. This is true for any linear operator $p(D)$.

The second operator is the first one multiplied by 2.

The equation

$$2x'' + 4x' + 4x = f(t)$$

can be rewritten as

$$x'' + 2x' + 2x = \frac{1}{2} f(t).$$

So the response of the system described by $2D^2 + 4D + 4I$ to any input $f(t)$ will be $1/2$ of the response of the system $D^2 + 2D + 2I$ to the same input. Applying this principle to the inputs $u(t)$ and $\delta(t)$ it follows that the step response and the unit impulse response get multiplied by $1/2$ when the operator gets multiplied by 2.

4. Using the time-invariance and your solution to part 3, write down a solution to

$$x'' + 2x' + 2x = 2u(t-1) + \delta(t-2) - 2u(t-3).$$

Time invariance gives the following information

Equation	Solution
$x'' + 2x' + 2x = u(t-1)$	$v(t-1)$
$x'' + 2x' + 2x = \delta(t-2)$	$w(t-2)$
$x'' + 2x' + 2x = u(t-3)$	$v(t-3)$

where v and w are the functions we determined in part 3. Hence a solution to our equation is

$$x = 2v(t-1) + w(t-2) - 2v(t-3),$$

that is,

$$x(t) = \begin{cases} 0 & t < 1 \\ 1 - e^{1-t}\cos(t-1) - e^{1-t}\sin(t-1) & 1 < t < 2 \\ \\ 1 - e^{1-t}\cos(t-1) - e^{1-t}\sin(t-1) + e^{2-t}\sin(t-2) & 2 < t < 3 \\ \\ -e^{1-t}\cos(t-1) - e^{1-t}\sin(t-1) + e^{2-t}\sin(t-2) & \\ +e^{3-t}\cos(t-3) + e^{3-t}\sin(t-3) & t > 3. \end{cases}$$