Convolution

$$f(t) * g(t) = \int_0^t f(t-\tau)g(\tau)d\tau$$

1. (a) What is the differential operator $p(D)$ whose weight function (i.e. unit impulse response) is the unit step function $u(t)$? Verify that $u(t) * q(t)$ is the solution to $p(D)x = q(t)$ with rest initial conditions.

   (b) What is the differential operator $p(D)$ whose weight function (i.e. unit impulse response) is $u(t)t$? Verify that $t * t^n$ is the solution to $p(D)x = t^n$ with rest initial conditions.

2. Solve $x' + x = u(t)(1 + \cos t)$ with rest initial conditions by computing the convolution product $w(t) * (1 + \cos t)$, where $w(t)$ is the unit impulse response of the operator $D + I$. Use the integral definition of convolution.

3. Now $p(D) = a_n D^n + \cdots + a_1 D + a_0 I$.

   (a) Suppose $a \geq 0$. Figure out what $w(t) * \delta(t-a)$ is by using the fact that is the solution to the equation $p(D)x = \delta(t-a)$ with rest initial conditions.

   (b) Then figure out what $w(t) * \delta(t-a)$ is by using the integral definition of convolution.