Convolution

$$f(t) * g(t) = \int_0^t f(t-\tau)g(\tau)d\tau$$

1. (a) What is the differential operator $p(D)$ whose weight function (i.e. unit impulse response) is the unit step function $u(t)$? Verify that $u(t) * q(t)$ is the solution to $p(D)x = q(t)$ with rest initial conditions.

   The answer to the first questions is just $p(D) = D$, since the solution of the equation $x' = \delta(t)$ is exactly $u(t)$.
   The equation $p(D)x = q(t)$ is then the same as $x' = q(t)$ with $x(0+) = 0$. It remains to check that the convolution $u(t) * q(t)$ satisfies this it.

$$\frac{d}{dt}u(t) * q(t) = \frac{d}{dt}\int_0^t u(t-\tau)q(\tau)d\tau = q(t) + \int_0^t -\delta(t-\tau)q(\tau)d\tau = q(t)$$

   (b) What is the differential operator $p(D)$ whose weight function (i.e. unit impulse response) is $u(t)t$? Verify that $t * t^n$ is the solution to $p(D)x = t^n$ with rest initial conditions.
   We want $w(t) = u(t)t$, so $w'(t) = u(t)$ and $w''(t) = \delta(t)$. Therefore $p(D) = D^2$.
   Now

$$\begin{aligned} t * t^n &= \int_0^t (t-\tau)\tau^n d\tau = t\int_0^t \tau^n d\tau - \int_0^t \tau^{n+1}d\tau = \\ &= t\cdot\frac{t^{n+1}}{n+1} - \frac{t^{n+2}}{n+2} = \frac{t^{n+2}}{(n+1)(n+2)} \end{aligned}$$

   and the second derivative of this function is clearly t^n.

2. Solve $x'+x = u(t)(1+\cos t)$ with rest initial conditions by computing the convolution product $w(t) * (1+\cos t)$, where $w(t)$ is the unit impulse response of the operator $D+I$. Use the integral definition of convolution.

First we need to compute $w(t)$, which means solving $x'+x = \delta(t)$. Here x' must have the same singularity as $\delta(t)$, which means that x must have a jump of 1 at $t=0$. So we want to solve

$$x' + x = 0 \quad \text{for } t > 0, \qquad x(0+) = 1.$$

The general solution is $x(t) = Ce^{-t}$ and the initial condition gives $C = 1$. So

$$w(t) = \begin{cases} e^{-t} & t > 0, \\ 0 & t < 0. \end{cases}$$

Then

$$\begin{aligned} w(t) * (1+\cos t) &= \int_0^t w(t-\tau)(1+\cos\tau)d\tau = \int_0^t e^{\tau - t}(1+\cos\tau)d\tau = \\ &= e^{-t}\int_0^t e^{\tau}(1+\cos\tau)d\tau \end{aligned}$$

Use integration by parts and your 18.02 prowess to do this last integral. The answer should be

3. Now $p(D) = a_n D^n + \cdots + a_1 D + a_0 I$.

   (a) Suppose $a \geq 0$. Figure out what $w(t) * \delta(t-a)$ is by using the fact that is the solution to the equation $p(D)x = \delta(t-a)$ with rest initial conditions.

   Time invariance says the solution is $w(t-a)$ for $t > a$ and 0 for $t < a$, so that's the result for our convolution.

(b) Then figure out what $w(t) * \delta(t-a)$ is by using the integral definition of convolution.

$$w(t) * \delta(t-a) = \int_0^t w(t-\tau)\delta(\tau-a)d\tau = \begin{cases} w(t-a) & t > a \\ 0 & t < a \end{cases}$$