MA 18.03, R05

Linear Systems I

1 Eigenvalues and eigenvectors of matrices

1. Given an $n \times n$ matrix A, how do you compute its determinant? How about the trace?

2. The characteristic polynomial of A is

   $p(\lambda) =$

3. The eigenvalues of A are

4. How many eigenvalues does A have, counting multiplicities?

5. An eigenvector corresponding to the eigenvalue λ of the matrix A is

6. Assume λ is a repeated eigenvalue of A with multiplicity m. When is it complete? When is it defective?

2 Systems of first order ODEs

What is an autonomous system?

The general solution of autonomous $n \times n$ linear system

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x}$$

is of the form

$$c_1\mathbf{x}_1(t) + \cdots + c_n\mathbf{x}_n(t),$$

where $c_1, \ldots, c_n$ are arbitrary constants and $\mathbf{x}_1(t), \ldots \mathbf{x}_n(t)$ are linearly independent solutions of the system of ODE's. To find the solution of an IVP, solve for constants.

1. How do we check that $\mathbf{x}_1(t), \ldots \mathbf{x}_n(t)$ are linearly independent?

2. To find $\mathbf{x}_1(t), \ldots \mathbf{x}_n(t)$ we proceed as follows:

   (a) Find the eigenvalues $\lambda_1, \ldots, \lambda_n$ of A.

   (b) For each of the **distinct** eigenvalues λ, there are a few possible cases:

   - If λ_j is a simple real eigenvalue, find an eigenvector $\mathbf{u}_j$. Then

     $$\mathbf{x}_j = \mathbf{u}_j e^{\lambda_j t}$$

     is the corresponding solution.
   - If λ_j and $\bar{\lambda}_j$ are simple complex conjugate eigenvalues of A, find an eigenvector $\mathbf{u}_j$ corresponding to λ_j. Then the two linearly independent solutions corresponding to λ_j and $\bar{\lambda}_j$ are given by

     $\mathbf{x}_j =$ $\qquad\qquad\qquad$ $\mathbf{x}_{j+1} =$

- If λ_j is a complete repeated eigenvalue with multiplicity m find m linearly independent eigenvectors $\mathbf{u}_1, \ldots, \mathbf{u_m}$ and use them to find the corresponding solutions according to the previous cases.
- If λ_j is a defective repeated eigenvalue, first try elimination. If that doesn't work find the generalized eigenvectors and ... (you tell me!)

3 2×2 Systems

Consider the autonomous linear system

$$\begin{cases} \dfrac{dx}{dt} &= ax + by \\ \dfrac{dy}{dt} &= cx + dy \end{cases}$$

- Review graphing systems.

- Consider the autonomous linear system

$$\begin{cases} \dfrac{dx}{dt} &= ax + by \\ \dfrac{dy}{dt} &= cx + dy \end{cases}$$

  1. Its matrix is

     $A =$

2. The critical point $(0, 0)$ (it will always be a critical point for a linear system!) can exhibit the following behaviors, according to the nature of the eigenvalues λ_1, λ_2 of A (fill in and draw):
   (a) If $\lambda_1, \lambda_2 < 0$ distinct real roots, then $(0, 0)$ is a ...

   (b) If $\lambda_1, \lambda_2 > 0$ distinct real roots, then $(0, 0)$ is a ...

   (c) If $\lambda_1 > 0, \lambda_2 < 0$ distinct real roots, then $(0, 0)$ is a ...

   (d) If $\lambda_1 = \lambda_2 < 0$ equal real roots, then $(0, 0)$ is a ...

   (e) If $\lambda_1 = \lambda_2 > 0$ equal real roots, then $(0, 0)$ is a ...

   (f) If λ_1, λ_2 are complex conjugates with the real part negative , then $(0, 0)$ is a ...

(g) If λ_1, λ_2 are complex conjugates with the real part positive , then $(0,0)$ is a ...

(h) If λ_1, λ_2 are complex conjugates with the real part 0 (i.e. purely imaginary), then $(0,0)$ is a ...

3. What happens if 0 is an eigenvalue?