1 Step and delta

Step function: $u(t) = \begin{cases} 1 & t > 0 \\ 0 & t < 0 \end{cases}$

This is the derivative of the *continuous* function $\begin{cases} t & t \geq 0 \\ 0 & t < 0. \end{cases}$

Notation: $u_a(t) = u(t-a)$ has a jump of 1 at the point $t = a$, is zero before that and 1 after.

Delta: the harpoon at 0 of size 1, is the derivative of $u(t)$. The harpoon indicates exactly the fact that its antiderivative has a jump of 1 at $t = 0$.

Notation: $\delta_a(t) = \delta(t-a)$ has a harpoon of size 1 at the point $t = a$ and is zero otherwise.

Properties:

- $\displaystyle\int_b^c f(t)\delta(t-a)dt = \begin{cases} f(a) & a \text{ is between } b, c \\ 0 & \text{otherwise} \end{cases}$
- $f(t)\delta(t-a) = f(a)\delta(t-a)$ harpoon of size $f(a)$ at $t = a$ and zero everywhere else.
- $u'(t-a) = \delta(t-a)$

2 Step and unit impulse response

Step response of an operator $p(D)$ is the solution to the equation $p(D)x = u(t)$. We usually denote it by $v(t)$.

In practice, it is found as follows for $p(D) = a_nD^n + \cdots + a_1D + a_0I$.

1. Solve the *inhomogeneous* equation $p(D)x = 1$ for $t > 0$ with the initial conditions (given by matching singularities) $x(0+) = x'(0+) = x^{(n-1)}(0+) = 0$.
2. Take the solution you found and multiply by the step function $u(t)$. This is your $v(t)$.

Note: The solution to $p(D)x = u_a(t)$ or $p(D)x = u(t-a)$ is $v(t-a)$.

Unit impulse response = weight function = fundamental solution of an operator $p(D)$ is the solution to the equation $p(D)x = \delta(t)$. We usually denote it by $w(t)$.

In practice, it is found as follows for $p(D) = a_nD^n + \cdots + a_1D + a_0I$.

1. Solve the *homogeneous* equation $p(D)x = 0$ for $t > 0$ with the initial conditions (given by matching singularities) $x^{(n-1)} = 1/a_n$ and $x(0+) = x'(0+) = x^{(n-2)}(0+) = 0$.
2. Take the solution you found and multiply by the step function $u(t)$. This is your $w(t)$.

Note: The solution to $p(D)x = \delta_a(t)$ or $p(D)x = \delta(t-a)$ is $w(t-a)$.

$$\boxed{w(t) = v'(t)}$$

MA 18.03, R05

3 Convolution

The convolution of two functions f and g is a function $(f * g)$ given by

$$(f * g)(x) = \int_0^x f(u)g(x-u)\,du.$$

Green's formula gives a way of finding a solution y_p to the IVP

$$ay'' + by' + cy = f(t), \qquad y(0) = y'(0) = 0$$

or more generally for an order n linear equation

$$p(D)y = f(t) \qquad y(0) = y'(0) = \ldots = y^{(n-1)}(0) = 0$$

in terms of the convolution of the input function $f(t)$ with a certain function associated to the system called the *unit impulse response or weight function*.

1. What is the *unit impulse response or weight function* for this ODE?

2. What is the *Green function* for this ODE?

3. Write down the formula for y_p in terms of the impulse response function (*Green's formula*).

4. What is the value of $y_p(0)$?

5. What is the value of $y_p'(0)$?