EXAM III REVIEW

1 Fourier Series

1.1 General facts

A (generalized) function $f(t)$ of period $2L$ has a Fourier series of the form

$$f(t) = \frac{a_0}{2} + a_1 \cos\frac{\pi t}{L} + a_2 \cos\frac{2\pi t}{L} + \cdots + b_1 \sin\frac{\pi t}{L} + b_2 \sin\frac{2\pi t}{L} + \cdots$$

1. How do we write this in $\sum$ notation?

2. What are the formulas for a_n and b_n?

   $a_n \quad =$

   $b_n \quad =$

3. What is the sum of the Fourier series? When is it convergent?

4. Are the following product of functions even or odd?
   - even · even =
   - even · odd =
   - odd· odd =

5. Write down all the sinusoidal functions of period 184.

Which ones are

(a) even?

(b) odd?

(c) even about 46?

(d) odd about 46?

6. Same question for a general period $2L$.

1.2 How do we compute Fourier series?

Please review the relevant materials and the worksheet I gave in class at the time (if you don't have it, look at `http://math.mit.edu/~alina/18.03`).

1.3 Use in ODE

Consider the linear ODE with constant coefficients

$$c_n y^{(n)} + \ldots + c_1 y' + c_0 y = f(t), \qquad c_n \neq 0, \quad f(t) \text{ periodic with period } 2L.$$

We can make use of Fourier series to find a **particular** solution y_p of this ODE, as follows.

1. First write down the Fourier series of $f(t)$.

2. Replace $f(t)$ by its Fourier series in the RHS of the ODE. It becomes something of the form

$$c_n y^{(n)} + \ldots + c_1 y' + c_0 y = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi t}{L} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi t}{L}.$$

3. For each term that appears in the Fourier series find a particular solution of the corresponding equation (use ERF or ERF' or ERF" etc. . .).

$$\begin{aligned}
c_n y^{(n)} + \ldots + c_1 y' + c_0 y &= \frac{a_0}{2} && \rightsquigarrow \text{ get } y_0 \\
c_n y^{(n)} + \ldots + c_1 y' + c_0 y &= a_n \cos\frac{n\pi t}{L} && \rightsquigarrow \text{ get } y_{1,n} \\
c_n y^{(n)} + \ldots + c_1 y' + c_0 y &= b_n \sin\frac{n\pi t}{L} && \rightsquigarrow \text{ get } y_{2,n}
\end{aligned}$$

4. A particular solution to the original ODE is the sum of all these pieces

$$y_p = y_0 + \sum_{n=1}^{\infty} y_{1,n} + \sum_{n=1}^{\infty} y_{2,n}.$$

Now look at the equation

$$y'' + \omega^2 y = f(t)$$

where $f(t)$ is the even function of period 2π that is equal to t for $0 < t < \pi$.

1. Write down the Fourier series of $f(t)$.

2. For which ω do we have resonance? How many terms give resonance for each of the values of ω?

3. Write down the *periodic* solution in the case of

   (a) resonance

   (b) non-resonance

4. Now write down the *general* solution in each of the above cases.

5. Now find a near-resonance problem and solve it.

2 Step and Delta

1. Write down the formula for the step function $u(t) =$

2. What's the relationship between $u(t)$ and $\delta(t)$?

3. Graph $u(t)$, $\delta(t)$, $f(t) = tu(t)$, $t\delta(t)$, $g(t) = (t^2 + 3)\delta_{-1}(t)$, $h(t) = (t^2 + 3)(u(t) - 2u(t-2) + u(t-3))$. Compute the $h'(t)$ using the product rule and graph it. Now do the same starting with the graph of $h(t)$.

4. $\displaystyle\int_a^b \delta(t-c)dt = \begin{cases} \\ \\ \end{cases}$

5. $\displaystyle\int_a^b f(t)\delta(t-c)dt = \begin{cases} \\ \\ \end{cases}$

6. $f(t)\delta(t-c) =$

3 Convolution and Green's Formula

The convolution of two functions f and g is a function $(f * g)$ given by

$$(f * g)(x) = \int_0^x f(u)g(x-u)\,du.$$

Green's formula gives a way of finding a solution y_p to the IVP

$$y'' + ay' + by = f(t), \qquad y(0) = y'(0) = 0$$

in terms of the convolution of the forcing function $f(t)$ with a certain function associated to the system called the *unit impulse response or weight function*.

1. What is the *step response* for this ODE?

2. What is the *unit impulse response or weight function* for this ODE?

3. What is the *Green function* for this ODE?

4. Write down the formula for y_p in terms of the impulse response function (*Green's formula*).

5. What is the value of $y_p(0)$?

6. What is the value of $y_p'(0)$?

7. Find the step response and weight function for the operator $p(D) = \frac{3}{2}(D^3 - 13D + 12I)$. Then write down a particular solution for the equation $p(D)x = f(t)$, where

$$f(t) = \begin{cases} 0 & t < 0 \\ e^t & 0 < t < 6 \\ e^{2t} & t > 6. \end{cases}$$

Finally, write down the general solution for $p(D)x = f(t)$.

4 Laplace Transform

If the function $f(t)$ is continuous at $t = 0$, there is no ambiguity about the way we define its Laplace transform:

$$\mathcal{L}(f)(s) = \int_0^\infty e^{-st} f(t)dt = F(s).$$

When we have to deal with discontinuities or generalized functions, it makes sense to consider

$$\mathcal{L}_+(f)(s) = \int_{0+}^\infty e^{-st} f(t)dt \quad \text{that ignores the (possible) discontinuity at 0 of } f(t);$$

$$\mathcal{L}_-(f)(s) = \int_{0-}^\infty e^{-st} f(t)dt \quad \text{that } \textbf{detects} \text{ if } f(t) \text{ has a discontinuity at 0.}$$

4.1 General facts

1. $af(t) + bg(t) \xrightarrow{\mathcal{L}} aF(s) + bG(s)$
2. $f(t) * g(t) \xrightarrow{\mathcal{L}} F(s)G(s)$
3. $e^{at} f(t) \xrightarrow{\mathcal{L}} F(s - a)$
4. $tf(t) \xrightarrow{\mathcal{L}} -F'(s)$
5. $t^n f(t) \xrightarrow{\mathcal{L}} (-1)^n F^{(n)}(s)$
6. $u(t - a)f(t - a) \xrightarrow{\mathcal{L}} e^{-as} F(s)$ for $a \geq 0$
7. $f'(t) \xrightarrow{\mathcal{L}} sF(s) - f(0)$ (use 0+ for $\mathcal{L}_+$)
8. $f''(t) \xrightarrow{\mathcal{L}} s^2 F(s) - sf(0) - f'(0)$ (use 0+ for $\mathcal{L}_+$)
9. $f^{(n)}(t) \xrightarrow{\mathcal{L}} s^n F(s) - s^{n-1} f(0) - \ldots - f^{(n-1)}(0)$ (use 0+ for $\mathcal{L}_+$)
10. $\mathcal{L}_-(f^{(n)}) = s^n F(s)$
11. Basic transforms

$$\mathcal{L}(u) = \mathcal{L}(1) = \frac{1}{s} \qquad \mathcal{L}(t^n) = \frac{n!}{s^{n+1}} \qquad \mathcal{L}(e^{at}) = \frac{1}{s - a}$$

$$\mathcal{L}(\cos at) = \frac{s}{s^2 + a^2} \qquad \mathcal{L}(\sin at) = \frac{a}{s^2 + a^2}$$

$$\mathcal{L}_-(\delta(t)) = 1 \qquad \mathcal{L}(\delta_a(t)) = e^{-as} \ (a > 0)$$

4.2 Use in ODE

Finding solutions of ODEs: take he Laplace transforms of both sides of an ODE, one obtains an (algeraic) equation for the Laplace transform. Solve that and take the inverse Laplace transform. That gives you a solution of the original ODE.

Finding the weight function of an operator $p(D)$: $\mathcal{L}(w(t)) = \dfrac{1}{p(s)}$

Pole diagram: the rightmost pole(s) of the Laplace transform $F(s)$ of a function $f(t)$ tells us how $f(t)$ behaves for very large t (as $t \to \infty$).

- If the right most pole is at $s = a$, then, for large t, $f(t) \sim$

- If the right most poles are at $s = a \pm ib$, then, for large t, $f(t) \sim$

- In particular, $f(t) \to 0$ as $t \to \infty$ if and only if the poles of $F(s)$ have what property?

- When is a system described by the operator $p(D)$ stable?

1. Find the Laplace transforms of $e^t \cos(3t) + 5e^{-t}$, $u(t) + \delta(t - 2\pi)$ and $t \sin t$.

2. Find the inverse Laplace transforms of $\dfrac{2s+3}{s^3+2s^2+s}$ and $\dfrac{2s+5}{s^2+4s+5}$.

3. Use the Laplace transform to compute $t^{21} * t^{125}$

4. Compute the Laplace transform of e^t first directly from the definition and then using the fact that

$$e^t = \sum_{n=0}^{\infty} \frac{t^n}{n!}.$$

5. Solve the initial value problem $x'' + 4x' - 5x = e^{-5t}$, $x(0) = 2$, $x'(0) = 3$.

6. Without solving, describe the long-term behavior of the solutions of the following ODEs. Decide which systems are stable and which are unstable.

   (a) $y'' + y' - 3y = \cos t$

   (b) $y'' + y' + 3y = \cos t$

(c) $y'' + y' + 3y = \sin t$

(d) $x'' + x' = 3e^{3t}$

(e) $x' + x = 3e^{-t}$

(f) $x'' + 2x' + x = 3e^{-t}$