1 Definition

If the function $f(t)$ is continuous, there is no ambiguity about the way we define its Laplace transform:

$$\mathcal{L}(f)(s) = \int_0^\infty e^{-st} f(t)dt = F(s).$$

When we have to deal with discontinuities or generalized functions, it makes sense to consider

$$\mathcal{L}_+(f)(s) = \int_{0+}^\infty e^{-st} f(t)dt \quad \text{that ignores the (possible) discontinuity at 0 of } f(t);$$

$$\mathcal{L}_-(f)(s) = \int_{0-}^\infty e^{-st} f(t)dt \quad \text{that } \textbf{detects} \text{ if } f(t) \text{ has a discontinuity at 0.}$$

2 Motivation

$$\begin{tikzcd}
\boxed{\text{ODE in } x(t) \text{ with initial conditions at } t=0} \arrow[r, "\mathcal{L}"] & \boxed{\text{algebraic equation in } X(s)} \arrow[d] \\
\boxed{x(t)} & \boxed{X(s)} \arrow[l, "\mathcal{L}^{-1}"']
\end{tikzcd}$$

3 Properties

They will be given in exams. Below $F(s)$ denotes the Laplace transform of $f(t)$ and $G(s)$ the Laplace transform of $g(t)$.

1. $af(t) + b(t) \xrightarrow{\mathcal{L}} aF(s) + bG(s)$
2. $e^{at} f(t) \xrightarrow{\mathcal{L}} F(s-a)$
3. $tf(t) \xrightarrow{\mathcal{L}} -F'(s)$
4. $u(t-a)f(t-a) \xrightarrow{\mathcal{L}} e^{-as}F(s)$ for $a \geq 0$
5. $f'(t) \xrightarrow{\mathcal{L}} sF(s) - f(0)$
6. $f''(t) \xrightarrow{\mathcal{L}} s^2F(s) - sf(0) - f'(0)$

4 Basic transforms

$$\mathcal{L}(1) = \frac{1}{s} \qquad \mathcal{L}(t^n) = \frac{n!}{s^{n+1}} \qquad \mathcal{L}(e^{at}) = \frac{1}{s-a}$$

$$\mathcal{L}(\cos at) = \frac{s}{s^2+a^2} \qquad \mathcal{L}(\sin at) = \frac{a}{s^2+a^2}$$

5 How to compute

Laplace transform: the idea is that you use the above properties and basic transforms to compute the Laplace transform of most functions. If you cannot figure it out this way, use the definition. But that should be your last resort.

Inverse Laplace transform: we don't have a direct method of computing it, what we do is identify the function of t whose Laplace transform the given function of s is. In practice, you break down your function as a linear combination of pieces that can be recognized as basic Laplace transforms using the properties in the above list.

6 How to use it to solve ODEs

1. Take the Laplace transform of both sides of your ODE (say the unknown function is $x(t)$). Do not forget to use the initial conditions (if given; if not given, find some that make your life easy).

2. Now you have an *algebraic* equation that involves $X(s)$ the Laplace transform of $x(t)$. Find $X(s)$.

3. Take the inverse Laplace transform of $X(s)$. That means identify the function of t whose Laplace transform is $X(s)$. This is your $x(t)$, the *solution to the initial value problem* you started with.
   So it is only a particular solution of the ODE, not the general solution. If you want the general solution, it is found by adding the homogeneous part to the guy you found.

7 The weight function revisited

Start with an operator $p(D) = a_n D^n + \ldots + a_1 D + a_0 I$. It has a weight function $w(t)$, defined to be the solution to the ODE

$$a_n x^{(n)} + \ldots a_1 x' + a_0 x = \delta(t).$$

$$\boxed{\mathcal{L}(w(t)) = \frac{1}{p(s)}}$$

So you can compute $w(t)$ by taking the inverse Laplace transform of $W(s) = \dfrac{1}{p(s)}$. The function $W(s)$ is called the *transfer function* of the operator $p(D)$.