LINEAR SYSTEMS

1 Eigenvalues and eigenvectors of matrices

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}$$

1. The trace of A is the sum of the elements on the diagonal, $\operatorname{Tr} A = a_{11} + a_{22} + \ldots + a_{nn}$.

2. The determinant of A is computed by expanding along a row or a column and keep doing it until we reduce the computation to 2×2 determinants. For instance, expanding along the i-th row we get

$$\det A = a_{i1}(-1)^{i+1} \det m_{i1} + a_{i2}(-1)^{i+2} \det m_{i2} + \ldots + a_{in}(-1)^{i+n} \det m_{in},$$

   where m_{ij} stands for the (i, j) minor of A, namely the $(n-1) \times (n-1)$ matrix obtained from A by erasing the i-th row and the j-th column.

3. A is invertible if and only if $\det A \neq 0$. In that case

$$A^{-1} = \frac{1}{\det A} \begin{bmatrix} m_{11} & -m_{12} & \dots & (-1)^{1+n} m_{1n} \\ -m_{21} & m_{22} & \dots & (-1)^{2+n} m_{2n} \\ \vdots & \vdots & & \vdots \\ (-1)^{n+1} m_{n1} & (-1)^{n+2} m_{n2} & \dots & m_{nn} \end{bmatrix}^T,$$

   where $[\ ]^T$ stands for the *transpose* of a matrix and the m_{ij}'s denote minors like above.

4. The characteristic polynomial of A is the degree n polynomial

$$p_A(\lambda) = \det(A - \lambda I_n).$$

5. The eigenvalues of A are the roots of $p_A(\lambda) = 0$. Counting multiplicities there are n of them.

6. An eigenvector corresponding to the eigenvalue λ of the matrix A is a **nonzero** vector $\mathbf{v}$ such that $A\mathbf{v} = \lambda\mathbf{v}$, or equivalently, $(A - \lambda I_n)\mathbf{v} = \mathbf{0}$.

7. Assume λ is a repeated eigenvalue of A with multiplicity m. It is called *complete* if there are m linearly independent eigenvectors corresponding to it and *defective* otherwise.

8. If λ is a defective eigenvalue and $\mathbf{v}$ is a corresponding eigenvector, a *generalized eigenvector* is a vector $\mathbf{u}$ such that $(A - \lambda I)\mathbf{u} = \mathbf{v}$.

2 Matrix exponentials

Definition The exponential of a square matrix A is $e^A = \sum_{k=0}^{\infty} \frac{1}{k!} A^k$.

Note:

- If B in another $n \times n$ matrix, then $e^{A+B} = e^A e^B$ if and only if $AB = BA$.
- If A has eigenvalues $\lambda_1, \ldots, \lambda_n$, the eigenvalues of its exponential e^A are $e^{\lambda_1}, \ldots, e^{\lambda_n}$.

How to compute

- Probably the easiest way is to make a system $\mathbf{x}' = A\mathbf{x}$ and find one of fundamental matrices $F(t)$. Then $e^A = F(1)F(0)^{-1}$.
- There is one other trick that might help with computations, namely if all the elements on the main diagonal are equal to r, write $A = rI + B$. Then B to some power gives the zero matrix and its exponential is easy to compute being a finite sum and $e^A = e^r e^B$.
- $D = \begin{bmatrix} a_1 & 0 & \ldots & 0 \\ 0 & a_2 & \ldots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \ldots & a_n \end{bmatrix}$ diagonal matrix $\Rightarrow e^D = \begin{bmatrix} e^{a_1} & 0 & \ldots & 0 \\ 0 & e^{a_2} & \ldots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \ldots & e^{a_n} \end{bmatrix}$.
- In general, you can diagonalize A, i.e. write it as $A = SDS^{-1}$ with D a diagonal matrix. Then $e^A = Se^D S^{-1}$.

3 Homogeneous linear systems of first order ODEs

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x}$$

The general solution of the system is of the form

$$\mathbf{x}(t) = c_1\mathbf{x}_1(t) + \cdots + c_n\mathbf{x}_n(t),$$

where $c_1, \ldots, c_n$ are arbitrary constants and $\mathbf{x}_1(t), \ldots \mathbf{x}_n(t)$ are linearly independent solutions of the system of ODE's. To find the solution of an IVP, solve for constants.

How to solve

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where $c_1, \ldots, c_n$ are arbitrary constants and $\mathbf{x}_1(t), \ldots \mathbf{x}_n(t)$ are linearly independent solutions of the system of ODE's. To find the solution of an IVP, solve for constants.
To find $\mathbf{x}_1(t), \ldots \mathbf{x}_n(t)$ we proceed as follows.

1. Find the eigenvalues $\lambda_1, \ldots, \lambda_n$ of A.

2. For each of the **distinct** eigenvalues λ, there are a few possible cases:

   - If λ_j is a simple real eigenvalue, find an eigenvector $\mathbf{u}_j$. Then
   $$\mathbf{x}_j = e^{\lambda_j t}\mathbf{u}_j$$
   is the corresponding solution. Such a solution is called a *normal mode* of the system.
   - If λ_j and $\lambda_{j+1} = \bar{\lambda}_j$ are simple complex conjugate eigenvalues of A, find a complex eigenvector $\mathbf{u}_j$ corresponding to λ_j. Then the two linearly independent solutions corresponding to λ_j and $\bar{\lambda}_j$ are given by
   $$\begin{aligned} \mathbf{x}_j &= \operatorname{Re}\left(e^{\lambda_j t}\mathbf{u}_j\right) \\ \mathbf{x}_{j+1} &= \operatorname{Im}\left(e^{\lambda_j t}\mathbf{u}_j\right). \end{aligned}$$
   - If λ_j is a complete repeated eigenvalue with multiplicity m find m linearly independent eigenvectors $\mathbf{u}_1, \ldots, \mathbf{u_m}$ and use them to find the corresponding solutions according to the previous cases.
   - If λ_j is a defective repeated eigenvalue, first try elimination. If that doesn't work and you really, really have to find the generalized eigenvectors and write down the corresponding solutions. Say that λ_j has multiplicity m and only one eigenvector $\mathbf{u}_j$. Then solve successively
   $$\begin{aligned} (A - \lambda_j I)\widehat{\mathbf{u}}_1 &= \mathbf{u}_j \\ (A - \lambda_j I)\widehat{\mathbf{u}}_2 &= \widehat{\mathbf{u}}_1 \\ &\vdots \\ (A - \lambda_j I)\widehat{\mathbf{u}}_{m-1} &= \widehat{\mathbf{u}}_{m-2} \end{aligned}$$
   The corresponding solutions are
   $$\begin{aligned} \mathbf{x}_j &= e^{\lambda_j t}\mathbf{u}_j \\ \mathbf{x}_{j+1} &= e^{\lambda_j t}(t\mathbf{u}_j + \widehat{\mathbf{u}}_1) \\ \mathbf{x}_{j+2} &= e^{\lambda_j t}(t^2\mathbf{u}_j + t\widehat{\mathbf{u}}_1 + \widehat{\mathbf{u}}_2) \\ &\vdots \\ \mathbf{x}_{j+m-1} &= e^{\lambda_j t}(t^{m-1}\mathbf{u}_j + \ldots + t\widehat{\mathbf{u}}_{m-2} + \widehat{\mathbf{u}}_{m-1}) \end{aligned}$$

Fundamental matrix

A linear system has *infinitely many* fundamental matrices. If $F(t)$ is any fundamental matrix, then
$$F'(t) = AF(t).$$
There is a distinguished one amongst them, denoted $F_0(t)$ in lecture. It has the following properties.

- $F_0(t) = e^{At}$.
- $F_0(0) = I_n$ the identity matrix.
- The solution to the IVP $\mathbf{x}' = A\mathbf{x}$, $\mathbf{x}(0) = \mathbf{v}$, is $\mathbf{x} = F_0(t)\mathbf{v}$.
- $F_0(1) = e^A$.

How to compute a fundamental matrix

1. Find n linearly independent solutions of the linear system $\mathbf{x}_1(t), \dots \mathbf{x}_n(t)$.
2. Write down the matrix $F(t)$ that has $\mathbf{x}_1(t), \dots \mathbf{x}_n(t)$ as columns. This is a fundamental matrix.

How to compute the distinguished fundamental matrix

Method I

1. First compute a fundamental matrix $F(t)$.
2. Compute $F(0)$ and take its inverse.
3. The distinguished fundamental matrix is $F_0(t) = F(t)\,(F(0))^{-1}$.

Method II

1. Find n linearly independent solutions of the linear system $\mathbf{x}_1(t), \dots \mathbf{x}_n(t)$.
2. Write down the general solution $\mathbf{x}(t) = c_1\mathbf{x}_1(t) + \cdots + c_n\mathbf{x}_n(t)$.
3. Find the solution $\mathbf{y}_j$ to the IVP $\mathbf{x}' = A\mathbf{x}, \quad \mathbf{x}(0) = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix}$, where 1 is on the j-th row.
4. The fundamental matrix $F(t)$ is the matrix with columns $\mathbf{y}_1, \dots, \mathbf{y}_n$.

Method III

Just compute the exponential e^{At} using either the series definition or, if possible, the trick of writing $At = rtI + Bt$. Then $F_0(t) = e^{At}$.

4 2×2 homogeneous linear systems

$$\begin{cases} \dfrac{dx}{dt} = ax + by \\[2ex] \dfrac{dy}{dt} = cx + dy \end{cases}$$

1. Its coefficient matrix is
$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$
2. The characteristic polynomial is $p(\lambda) = \lambda^2 - (\operatorname{Tr} A)\lambda + \det A$.
3. The critical point $(0,0)$ (it will always be a critical point for a linear system!) can exhibit the following behaviors, according to the nature of the eigenvalues λ_1, λ_2 of A:
   - If $\lambda_1 < \lambda_2 < 0$ distinct real roots, then $(0,0)$ is an improper nodal sink (stable). The solutions will be of the form $\mathbf{x} = c_1 e^{\lambda_1 t}\mathbf{v}_1 + c_2 e^{\lambda_2 t}\mathbf{v}_2$, where $\mathbf{v}_j$ is an eigenvector cor-

responding to the eigenvalue λ_j. The trajectories will be tangent to the line spanned by $\mathbf{v}_2$ and follow the direction of $\mathbf{v}_1$. They all approach the origin.

- If $\lambda_1 > \lambda_2 > 0$ distinct real roots, then $(0,0)$ is an improper nodal source (unstable). The solutions will be of the form $\mathbf{x} = c_1 e^{\lambda_1 t}\mathbf{v}_1 + c_2 e^{\lambda_2 t}\mathbf{v}_2$, where $\mathbf{v}_j$ is an eigenvector corresponding to the eigenvalue λ_j. The trajectories will be tangent to the line spanned by $\mathbf{v}_2$ and follow the direction of $\mathbf{v}_1$. They all go away from the origin.

- If $\lambda_1 > 0, \lambda_2 < 0$ distinct real roots, then $(0,0)$ is a saddle (unstable). The solutions will be of the form $\mathbf{x} = c_1 e^{\lambda_1 t}\mathbf{v}_1 + c_2 e^{\lambda_2 t}\mathbf{v}_2$, where $\mathbf{v}_j$ is an eigenvector corresponding to the eigenvalue λ_j.

- If $\lambda_1 = \lambda_2 = \lambda < 0$ equal real negative roots and λ is a complete eigenvalue with linearly independent eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$, then $(0,0)$ is a proper nodal sink (star-shaped, stable). The general solution is $\mathbf{x} = c_1 e^{\lambda t}\mathbf{v}_1 + c_2 e^{\lambda t}\mathbf{v}_2$.

- If $\lambda_1 = \lambda_2 = \lambda > 0$ equal real positive roots and λ is a complete eigenvalue with linearly independent eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$, then $(0,0)$ is a proper nodal source (star-shaped, unstable). The general solution is $\mathbf{x} = c_1 e^{\lambda t}\mathbf{v}_1 + c_2 e^{\lambda t}\mathbf{v}_2$.

- If $\lambda_1 = \lambda_2 = \lambda < 0$ equal real negative roots and λ is a defective eigenvalue with eigenvector $\mathbf{v}_1$, then $(0, 0)$ is a proper nodal sink (stable). The general solution is $\mathbf{x} = c_1 e^{\lambda t}\mathbf{v}_1 + c_2 e^{\lambda t}(t\mathbf{v}_1 + \mathbf{v}_2)$, where $\mathbf{v}_2$ is a generalized eigenvector. All trajectories are tangent to the line spanned by the eigenvector $\mathbf{v}_1$.

- If $\lambda_1 = \lambda_2 = \lambda > 0$ equal real positive roots and λ is a defective eigenvalue with eigenvector $\mathbf{v}_1$, then $(0, 0)$ is a proper nodal source (unstable). The general solution is $\mathbf{x} = c_1 e^{\lambda t}\mathbf{v}_1 + c_2 e^{\lambda t}(t\mathbf{v}_1 + \mathbf{v}_2)$, where $\mathbf{v}_2$ is a generalized eigenvector. All trajectories are tangent to the line spanned by the eigenvector $\mathbf{v}_1$.

- If $\lambda_1, \lambda_2 = \alpha \pm i\beta$ are complex conjugates with the real part negative , then $(0, 0)$ is a spiral sink (stable). The direction of the spiral will be counterclockwise if $b > 0$ and $c < 0$ and clockwise if $b < 0$ and $c > 0$. *Note that this is condition on the coefficient matrix!*

  The general solution will be $\mathbf{x} = c_1 \operatorname{Re}(e^{\lambda t}\mathbf{v}) + c_2 \operatorname{Im}(e^{\lambda t}\mathbf{v})$.

- If $\lambda_1, \lambda_2 = \alpha \pm i\beta$ are complex conjugates with the real part positive , then $(0, 0)$ is a spiral source (unstable). As in the previous case, the direction of the spiral will be counterclockwise if $b > 0$ and $c < 0$ and clockwise if $b < 0$ and $c > 0$.

- If $\lambda_1, \lambda_2 = \pm i\beta$ are complex conjugates with the real part 0 (i.e. purely imaginary), then $(0, 0)$ is a stable center. The trajectories look like ellipses centered at the origin. Again, the direction of the trajectories will be counterclockwise if $b > 0$ and $c < 0$ and clockwise if $b < 0$ and $c > 0$.

- If $\lambda_1 < \lambda_2 = 0$, and $\mathbf{v}_j$ is an eigenvector corresponding to the eigenvalue λ_j, $j = 1, 2$, the general solution is $\mathbf{x} = c_1 e^{\lambda_1 t}\mathbf{v}_1 + c_2\mathbf{v}_2$. The trajectories on the line spanned by $\mathbf{v}_2$ are just points. The rest of the trajectories are parallel to $\mathbf{v}_1$. The phase portrait looks like

- If $\lambda_1 > \lambda_2 = 0$, and $\mathbf{v}_j$ is an eigenvector corresponding to the eigenvalue λ_j, $j = 1, 2$, the general solution is $\mathbf{x} = c_1 e^{\lambda_1 t}\mathbf{v}_1 + c_2\mathbf{v}_2$. The trajectories on the line spanned by $\mathbf{v}_2$ are just points. The rest of the trajectories are parallel to $\mathbf{v}_1$. The critical point $(0, 0)$ is unstable and the phase portrait looks like

- If $\lambda_1 = \lambda_2 = 0$ is a repeated complete eigenvalue, all trajectories are just points in the plane. Notice that in this case the matrix A is just the zero matrix and all solutions are constant.

- If $\lambda_1 = \lambda_2 = 0$ is a repeated defective eigenvalue with eigenvector $\mathbf{v}_1$, the general solution is $\mathbf{x} = c_1\mathbf{v}_1 + c_2\left(t\mathbf{v}_1 + \mathbf{v}_2\right)$. The trajectories are lines parallel to $\mathbf{v}_1$.

5 Non-homogeneous $n \times n$ linear systems of ODEs

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x}(t) + \mathbf{b}(t)$$

Once again the general solution is of the form $\mathbf{x} = \mathbf{x}_p + \mathbf{x}_h$, with $\mathbf{x}_p$ a particular solution of the non-homogeneous system and $\mathbf{x}_h$ the solution to the corresponding homogenous system $\mathbf{x}' = A\mathbf{x}$.

How to solve

1. Guess by whatever means a particular solution $\mathbf{x}_p$, compute $\mathbf{x}_h$ by the previous algorithm and add them up.
2. **Variation of parameters:**
   - Compute the distinguished fundamental matrix $F_0(t) = e^{tA}$. Or, equivalently, compute the exponential e^{tA} directly.
   - The general solution is of the form

$$\boxed{\mathbf{x}(t) = \int_0^t e^{(t-\tau)A}\mathbf{b}(\tau)d\tau + e^{tA}\mathbf{v}}$$

   where $\mathbf{v}$ is any n-dimensional vector (=your arbitrary constants!). Note that $\mathbf{x}(0) = \mathbf{v}$, so IVPs are really easy to solve.