FIRST ORDER DIFFERENTIAL EQUATIONS

1. A first order differential equation is an equation of the form

$$F(x, y, y') = 0.$$

   A solution of the differential equation is a function $y = y(x)$ that satisfies the equation. A differential equation has **infinitely many** solutions.

2. An equilibrium solution is a constant solution, i.e. a constant function $y(x) = K$ that satisfies the equation.

3. An initial value problem consists of a differential equation and an initial condition $y(x_0) = y_0$. It has a **unique** solution, which has to be a function (provided F doesn't have problems at that point). For instance $y = \pm\sqrt{x}$ cannot be the solution of an i.v.p.

A. Separable Equations

$$\frac{dy}{dx} = f(x)g(y)$$

1. Separate the variables: $\dfrac{dy}{g(y)} = f(x)dx$.
2. Integrate both sides: $\displaystyle\int \frac{dy}{g(y)} = \int f(x)dx$ and get $G(y) = C + F(x)$.
3. Solve for y (if possible).
4. If it is an initial value problem, plug in the given values and solve for C.

B. Linear equations

$$\frac{dy}{dx} + P(x)y = Q(x)$$

1. Write the equation in the above form.
2. Compute the integrating factor $I(x) = e^{\int P(x)dx}$.
3. Multiply both sides of the equation by this integrating factor: $I(x)\dfrac{dy}{dx} + I(x) \cdot P(x)y = I(x) \cdot Q(x)$.
4. Integrate both sides and get $I(x)y = C + \int I(x)Q(x)dx$.
5. Solve for y.
6. If it is an initial value problem, plug in the given values and solve for C.

C. Homogeneous Equations

$$\frac{dy}{dx} = F\left(\frac{y}{x}\right)$$

1. Make the substitution $v = \dfrac{y}{x}$. **Attention:** $\dfrac{dy}{dx} = x \cdot \dfrac{dv}{dx} + v$.
2. Solve the new equation, not forgetting the constant involved. (It will probably be a *separable equation.*)
3. Find $y(x) = xv(x)$.
4. If it is an initial value problem, make sure you find the constant.

D. Bernoulli Equations

$$\frac{dy}{dx} + P(x)y = Q(x)y^n$$

1. Make the substitution $v = y^{1-n}$.
2. Solve the new *linear* equation: $\dfrac{dv}{dx} + (1-n)P(x)y = (1-n)Q(x)$.
3. Find $y(x)$.
4. If it is an initial value problem, make sure you find the constant.

E. Other Substitutions

Some equations need some other substitution to transform them in a known type. These substitutions vary greatly and there are no general formulas that can help. However, the more you practice, the better your intuition becomes in these matters.

F. Linear first order equations with sinusoidal input

$$y' + ky = B\cos(\omega t) \text{ or } B\sin(\omega t) \qquad (*)$$

Such an equation can be seen as the real or imaginary part of the complex differential equation

$$z' + kz = Be^{i\omega t} \qquad (**)$$

The general idea is that the solutions to $(*)$ are of the form

$$y = y_p + y_h,$$

where y_p is a *particular* solution of the original equation $(*)$, and y_h is the *general* solution to the associated homogeneous equation (see below).

1. Solve the corresponding *homogeneous equation* $y' + ky = 0$. The solution is $y_h = Ce^{-kt}$. *This is where the arbitrary constant appears!*
2. Find a particular solution z_p of $(**)$. It will usually be of the same form as the input, namely$Ae^{i\omega t}$. So try that and determine A.
3. Write y_p by taking either the real or imaginary part of z_p. If the input in $(*)$ is cos, you should take the real part, if it is sin, the imaginary part.
4. Write down the general solution to $(*)$, $y = y_p + y_h$
5. If it is an initial value problem, find the constant.

G. General strategy

If you are asked to solve a first order ODE and no method is specified, go through the following steps in order to determine what method to apply.

1. Is it separable?
2. Is it linear? Does it have sinusoidal or exponential input?
3. Is it homogeneous? Reducible?
4. Is it a Bernoulli equation?
5. Can you think of a substitution that would put it in a form you recognize?

H. (Simple) Euler's method

In order to apply the SE method with step of size h starting at the point (x_0, y_0) and ending at the point (x_N, y_N) for the ODE

$$y' = f(x, y)$$

you'll need to fill out the following table.

n	x_n	y_n	A_n	hA_n
0	x_0	y_0	$f(x_0, y_0)$	hA_0
1	$x_1 = x_0 + h$	$y_1 = y_0 + hA_0$	$f(x_1, y_1)$	hA_1
$\vdots$				
N	$x_N = x_{N-1} + h$	$y_N = y_{N-1} + hA_{N-1}$		

The relevant formulas are

$$x_n = x_{n-1} + h, \qquad A_n = f(x_n, y_n), \qquad y_n = y_{n-1} + hA_{n-1}.$$

The value you obtain for y_N will approximate the solution to the given ODE that satisfies the initial conditon $y(x_0) = y_0$. The approximation will be too high if the solution $y(x)$ is concave down and too low if the solution is concave up. In order to determine if y is concave up or down, you need to compute y'' (chain rule and product rule!) starting with the fact that $y' = f(x, y)$.

I. Graphing, direction fields, equilibrium points, phase line, bifurcation diagram

Lectures 1 and 8.

J. Sub and supersolutions, linear vs nonlinear

Lecture 9.