SECOND ORDER LINEAR HOMOGENEOUS ODEs

$$mx'' + bx' + kx = 0, \quad m, b, k \text{ constants, } m \neq 0$$

1. If the equation describes a physical spring–dashpot system, the coefficients m, b, k are non-negative.
2. An initial value problem consists of a differential equation and an initial condition $x(t_0) = A,\ x'(t_0) = B$. How many solutions does it have?
3. A boundary problem consists of a differential equation and a boundary condition $x(t_0) = A,\ x(t_1) = B$. How many solutions does it have?
4. The general solution is of the form $x(t) = c_1x_1(t)+c_2x_2(t)$, where x_1 and x_2 are two *linearly independent* solutions (none of them can be written as a constant multiple of the other).
5. The characteristic polynomial of this equations is $p(s) = ms^2 + bs + k$.
6. The exponential solutions of this equation are $c_1e^{r_1t}$ and $c_2e^{r_2t}$, where r_1, r_2 are the roots (real or complex) of the characteristic polynomial and c_1, c_2 are arbitrary constants. If $r_1 = r_2 = r$, there is only one family of exponential solutions, namely ce^{rt}.

How to solve:

1. Write down the characteristic equation $ms^2 + bs + k = 0$.
2. Compute its discriminant $\Delta = b^2 - 4mk$.
3. There are three possible situations:
   - *overdamped:* If $\Delta > 0$, the quadratic equation has two distinct real solutions, r_1 and r_2. Find them. (You might need to use the quadratic formula.) The general solution of the differential equation is
   $$x = C_1e^{r_1t} + C_2e^{r_2t}.$$
   - *critically damped:* If $\Delta = 0$, the quadratic equation has only one real root r. Find it. The general solution of the differential equation is
   $$x = C_1e^{rt} + C_2te^{rt}.$$
   - *underdamped:* If $\Delta < 0$, the quadratic equation does not have any real roots, it has two complex conjugate roots $r_{1,2} = \alpha \pm i\beta$, where $\alpha = -\frac{b}{2m}$ and $\beta = \frac{\sqrt{|\Delta|}}{2m}$. The general solution of the differential equation is
   $$x = C_1e^{\alpha t}\cos(\beta t) + C_2e^{\alpha t}\sin(\beta t) \qquad \text{or} \qquad x = Ae^{\alpha t}\cos(\beta t - \phi).$$
   In the second expression, A is any real number and ϕ any number in $[0, 2\pi)$.
4. If it is an initial value problem or a boundary problem, plug in the given values and solve for C_1 and C_2 (or for A and ϕ). Don't forget to take the derivative of x in the case of an initial value problem (chain rule!).