SECOND ORDER ODE'S

1. A second order differential equation is an equation of the form

$$F(x, y, y', y'') = 0.$$

   A solution of the differential equation is a function $y = y(x)$ that satisfies the equation. A differential equation has **infinitely many** solutions. They usually involve two undetermined constants.

2. An initial value problem consists of a differential equation and an initial conditions $y(x_0) = y_0, y'(x_0) = y'_0$. It has a **unique** solution, which has to be a function (provided F doesn't have problems at that point).

3. A boundary problem consists of a differential equation and conditions

$$y(x_0) = y_0, \quad y(x_1) = y_1;$$

   or

$$y'(x_0) = y'_0, \quad y(x_1) = y_1;$$

   or

$$y(x_0) = y_0, \quad y'(x_1) = y'_1;$$

   or

$$y'(x_0) = y'_0, \quad y'(x_1) = y'_1.$$

A. Homogeneous linear second order ODEs with constant coefficients

$$mx'' + bx' + kx = 0, \quad m, b, k \text{ constants}, m \neq 0$$

1. If the equation describes a physical spring–dashpot system, the coefficients m, b, k are non-negative.

2. An initial value problem consists of a differential equation and an initial condition $x(t_0) = A,\ x'(t_0) = B$. How many solutions does it have?

3. The general solution is of the form $x(t) = c_1 x_1(t) + c_2 x_2(t)$, where x_1 and x_2 are two *linearly independent* solutions (none of them can be written as a constant multiple of the other).

4. The characteristic polynomial of this equations is $p(s) = ms^2 + bs + k$.

5. The exponential solutions of this equation are $c_1 e^{r_1 t}$ and $c_2 e^{r_2 t}$, where r_1, r_2 are the roots (real or complex) of the characteristic polynomial and c_1, c_2 are arbitrary constants. If $r_1 = r_2 = r$, there is only one family of exponential solutions, namely ce^{rt}.

How to solve:

1. Write down the characteristic equation $ms^2 + bs + k = 0$.

2. Compute its discriminant $\Delta = b^2 - 4mk$.

3. There are three possible situations:

- *overdamped:* If $\Delta > 0$, the quadratic equation has two distinct real solutions, r_1 and r_2. Find them. (You might need to use the quadratic formula.) The general solution of the differential equation is

$$x = C_1 e^{r_1 t} + C_2 e^{r_2 t}.$$

- *critically damped:* If $\Delta = 0$, the quadratic equation has only one real root r. Find it. The general solution of the differential equation is

$$x = C_1 e^{rt} + C_2 t e^{rt}.$$

- *underdamped:* If $\Delta < 0$, the quadratic equation does not have any real roots, it has two complex conjugate roots $r_{1,2} = \alpha \pm i\beta$, where $\alpha = -\frac{b}{2m}$ and $\beta = \frac{\sqrt{|\Delta|}}{2m}$. The general solution of the differential equation is

$$x = C_1 e^{\alpha t}\cos(\beta t) + C_2 e^{\alpha t}\sin(\beta t) \qquad \text{or} \qquad x = Ae^{\alpha t}\cos(\beta t - \phi).$$

  In the second expression, A is any positive real number and ϕ any number in $[0, 2\pi)$.

4. If it is an initial value problem or a boundary problem, plug in the given values and solve for C_1 and C_2 (or for A and ϕ). Don't forget to take the derivative of x in the case of an initial value problem (chain rule!).

B. Nonhomogeneous second order linear ODEs

$$mx'' + bx' + kx = f(t).$$

We solve these by finding a *particular* solution x_p of the given ODE and the solution x_h to the corresponding homogeneous equation $mx'' + bx' + kx = 0$. The general solution is given by $x = x_p + x_h$.

Note: x_h contains all the undetermined constants (denotes an infinite family of functions), while x_p is **one** particular function.

We have two ways of finding x_p, namely ERF and undetermined coefficients. They are both explained below for linear equations of arbitrary order. Just apply them for your second order ODE.

LINEAR ODEs AND LINEAR OPERATORS

A. Linear operators

Notation

- D stands for the derivative of a function. For instance, if x is a function of t, $Dx = x' = dx/dt$. If f is a function of x, then $Df = f' = df/dx$.
- D^n stands for taking the n-th derivative. For instance $D^2x = D(Dx) = D(x') = x''$.
- I is the identity operator that leaves functions alone, so $I(f) = f$.
- In general $p(D) = a_nD^n + \ldots + a_1D + a_0I$ applied to a function y means taking a linear combination of y and its first n derivatives, namely

$$p(D)y = (a_nD^n + \ldots + a_1D + a_0I)y = a_ny^{(n)} + \ldots + a_1y' + a_0y$$

  Writing $p(D)y = f(t)$ provides an handy shorthand for the linear ODE

$$a_ny^{(n)} + \ldots + a_1y' + a_0y = f(t)$$

Properties

- *Linearity*
$$p(D)(af + bg) = ap(D)f + bp(D)g$$
- *Exponential shift formula*
$$p(D)(e^{rt}u) = e^{rt}p(D + rI)u$$

Don't forget that $(D + rI)^2 = D^2 + 2rD + r^2I$ (and so forth).

B. Homogeneous linear ODEs with constant coefficients

$$a_n y^{(n)} + \ldots + a_1 y' + a_0 y = 0$$

This equation can be re-written as
$$p(D)y = 0$$
with
$$p(D) = a_n D^n + \ldots + a_1 D + a_0 I$$

1. Write down the characteristic polynomial $p(s) = a_n s^n + \ldots + a_1 s + a_0$.
2. Find its n roots $r_1, \ldots, r_n$ (counted with multiplicity) by solving $a_n r^n + \ldots + a_0 = 0$.
3. The general solution is of the form $x = c_1 x_1 + \ldots + c_n x_n$ where each x_j corresponds to the root r_j as follows.
   - For each simple real root r_j we obtain an exponential solution $x_j = e^{r_j t}$.
   - For each pari of simple complex conjugate roots $r_{j,j+1} = a \pm ib$ we get the pair of solutions$x_j = e^{at} \cos bt$ and $x_{j+1} = e^{at} \sin bt$
   - A repeated real root r_j with multiplicity m, should have m corresponding solutions. Construct x_j as above. The rest are $tx_j, t^2 x_j, \ldots, t^{m-1} x_j$.
   - For repeated pairs of complex roots the same principle applies, except now you have m pairs of corresponding solutions.

C. Non-homogeneous linear ODEs

$$a_n y^{(n)} + \ldots + a_1 y' + a_0 y = 0$$

This equation can be re-written as
$$p(D)y = 0$$
with
$$p(D) = a_n D^n + \ldots + a_1 D + a_0 I$$

We solve these by finding a *particular* solution x_p of the given ODE and the solution x_h to the corresponding homogeneous equation $mx'' + bx' + kx = 0$. The general solution is given by $x = x_p + x_h$.

Note: x_h contains all the undetermined constants (denotes an infinite family of functions), while x_p is **one** particular function.

We have two ways of finding x_p, detailed below.

Exponential response formulas: can be applied to an equation with exponential input

$$p(D)x = Ae^{kt},$$

where r is any complex number.

1. Write down the characteristic polynomial $p(s)$.
2. Compute $p(k)$. If it is nonzero, then $x_p = \frac{Ae^{kt}}{p(k)}$ is a solution. (ERF)
3. If $p(k) = 0$, compute $p'(k)$. If this in nonzero, then $x_p = \frac{Ate^{kt}}{p'(k)}$ is a solution. (ERF')
4. Keep going. The idea is to compute $p(k), p'(k), p''(k), \ldots$ until you find the first nonzero guy amongst them. Say that happens to be $p^{(m)}(k)$. Then

$$x_p = \frac{At^m e^{kt}}{p^{(m)}(k)}$$

is a solution.

Undetermined coefficients method: the idea is to look for a solution of the same general form as the function $f(t)$ on the RHS of our equation.

If $f(t)$ is of the form	try
ae^{rt}	Ae^{rt}
$a\cos\omega t$	$A\cos\omega t + B\sin\omega t$
$a\sin\omega t$	$A\cos\omega t + B\sin\omega t$
$a_nt^n + \ldots + a_0$	$A_nt^n + \ldots + A_0$
$(a_nt^n + \ldots + a_0)e^{rt}$	$(A_nt^n + \ldots + A_0)e^{rt}$
$(a_nt^n + \ldots + a_0)\cos\omega t$	$(A_nt^n + \ldots + A_0)(B_1\cos\omega t + B_2\sin\omega t)$
$(a_nt^n + \ldots + a_0)\sin\omega t$	$(A_nt^n + \ldots + A_0)(B_1\cos\omega t + B_2\sin\omega t)$
$(a_nt^n + \ldots + a_0)e^{rt}\cos\omega t$	$(A_nt^n + \ldots + A_0)e^{rt}(B_1\cos\omega t + B_2\sin\omega t)$
$(a_nt^n + \ldots + a_0)e^{rt}\sin\omega t$	$(A_nt^n + \ldots + A_0)e^{rt}(B_1\cos\omega t + B_2\sin\omega t)$

If $f(t) = f_1(t) + f_2(t)$, look for solutions $x_j(t)$ corresponding to $f_j(t)$, $j = 1, 2$, and then add them up to get $x_p = x_1 + x_2$.

G. Physical notions

complex gain
amplitude gain

phase lag/shift
time lag
amplitude
resonance
practical resonance
frequency
pseudofrequency