Non-homogeneous linear ODEs with constant coefficients

$$a_\ell y^{(\ell)} + a_{\ell-1} y^{(\ell-1)} + \cdots + a_1 y' + a_0 y = G(x), \text{ where } a_j \text{ are real numbers, } a_\ell \neq 0$$

Note We can solve this only when $G(x)$ looks like $P(x)e^{kx}\cos(mx)$ or $P(x)e^{kx}\sin(mx)$, where P is a polynomial in x (could be 0 or 1 or any other constant), k is any real number (including 0) and m is any integer (this could be 0 as well). In case $m = 0$, $G(x) = P(x)e^{kx}$.

1. Write down the associated homogeneous equation

$$a_\ell y^{(\ell)} + a_{\ell-1} y^{(\ell-1)} + \cdots + a_1 y' + a_0 y = 0$$

and find its general solution y_h, as follows.

(a) Write down the characteristic polynomial $p(s) = a_\ell s^\ell + \cdots + a_0$.

(b) Find its roots $r_1, \ldots, r_\ell$.

(c) Write down the general solution to the homogeneous equation

$$y_h = C_1 y_1 + \cdots + C_\ell y_\ell.$$

Take into account multiplicities (double/triple/etc... roots) and pairs of complex roots.

2. Find a particular solution y_p of the non-homogeneous equation using the **method of undetermined coefficients**.

If $G(x) = P(x)e^{kx}\cos(mx)$ or $G(x) = P(x)e^{kx}\sin(mx)$, try

$$y_p = x^s \Big[R_1(x) e^{kx} \cos(mx) + R_2 e^{kx} \sin(mx) \Big],$$

where $\deg(R_1) = \deg(R_2) = \deg(P)$ and s is the smallest nonnegative integer such that no term in y_p duplicates a term in y_h. For instance,

- if $G(x) = de^{kx}$, then try $y_p = Ax^s e^{kx}$;
- if $G(x) = P(x)e^{kx}$, then try $y_p = x^s R(x) e^{kx}$ with R a polynomial in x of the same degree as P;
- if $G(x) = P(x)$, then try $y_p = x^s R(x)$ with R a polynomial in x of the same degree as P;
- if $G(x) = d\cos(mx)$ or $G(x) = d\sin(mx)$ or $G(x) = d_1\cos(mx) + d_2\sin(mx)$, then try $y_p = x^s\left(A\cos(mx) + B\sin(mx)\right)$;
- if $G(x) = e^{kx}\left(d_1\cos(mx) + d_2\sin(mx)\right)$, then try $y_p = x^s e^{kx}\left(A\cos(mx) + B\sin(mx)\right)$;
- if $G(x) = P(x)\cos(mx)$ or $G(x) = P(x)\sin(mx)$ or $G(x) = P(x)\left(d_1\cos(mx) + d_2\sin(mx)\right)$, then try $y_p = x^s\left(R_1(x)e^{kx}\cos(mx) + R_2 e^{kx}\sin(mx)\right)$, with R_1 and R_2 polynomials in x of the same degree as P.

If $G(x) = G_1(x) + G_2(x)$, then find $\tilde{y}_j$, $j = 1, 2$ particular solution for the equation

$$a_\ell y^{(\ell)} + a_{\ell-1} y^{(\ell-1)} + \cdots + a_1 y' + a_0 y = G_j(x)$$

and then take $y_p = \tilde{y}_1 + \tilde{y}_2$.

3. Write down the general solution $y = y_h + y_p$.

4. If it is an initial value problem or a boundary problem, plug in the given values and solve for $C_1, \ldots, C_\ell$. Don't forget to take the derivative of y in the case of an initial value problem (chain rule!).