HOMEWORK 4

DUE 7 OCTOBER 2008

This Pset works through the proof that the weight 2 Eisenstein series is a *quasi-modular* form.

Define $G_2 : \mathcal{H} \to \mathbb{C}$ by

$$G_2(\tau) = 2\zeta(2) + 2(2\pi i)^2 \sum_{n=1}^{\infty} \sigma_1(n) q^n,$$

where $q = e^{2\pi i \tau}$. Note that the series is absolutely convergent for $|q| < 1$, i.e. for $\Im(\tau) > 0$. Hence G_2 is analytic on $\mathcal{H}$.

1. Show that $G_2(\tau + 1) = G_2(\tau)$ (this is the transformation under T).

2. To see how G_2 transforms under the action of the other generator, S, start by showing that

$$G_2(\tau) = 2\zeta(2) + \sum_{n \in \mathbb{Z}\setminus\{0\}} \sum_{m \in \mathbb{Z}} \frac{1}{(m + n\tau)^2}$$

3. Now prove that

$$\frac{1}{\tau^2} G_2\left(-\frac{1}{\tau}\right) = 2\zeta(2) + \sum_{m \in \mathbb{Z}} \sum_{n \in \mathbb{Z}\setminus\{0\}} \frac{1}{(m + n\tau)^2}.$$

Be careful with the order of summation!

4. Show that

$$\sum_{n \in \mathbb{Z}\setminus\{0\}} \frac{1}{(m + n\tau)^2} = -8\pi^2 \int_0^{\infty} \cos(2\pi m t) g_\tau(t) dt,$$

where $g_\tau(t) = t \sum_{n=1}^{\infty} e^{2\pi i n t \tau}$ for $t > 0$

and $g_\tau(0) = \lim_{t \to 0+} g_\tau(t) = -\frac{1}{2\pi i \tau}$.

5. Now use the previous point to prove that

$$\sum_{m \in \mathbb{Z}} \sum_{n \in \mathbb{Z}\setminus\{0\}} \frac{1}{(m + n\tau)^2} = \sum_{n \in \mathbb{Z}\setminus\{0\}} \sum_{m \in \mathbb{Z}} \frac{1}{(m + n\tau)^2} - \frac{2\pi i}{\tau}.$$

Note: In particular this shows that the double sum is **not** absolutely convergent.

6. Prove that G_2 transforms under the action of $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ as follows:

$$G_2\left(-\frac{1}{\tau}\right) = \tau^2 G_2(\tau) - 2\pi i \tau.$$

7. Quasi-modularity: Prove that for any matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z})$ and any $\tau \in \mathcal{H}$ we have

$$G_2\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^2 G_2(\tau) - 2\pi i c(c\tau + d).$$

Note: You have to show that $c(c\tau + d)$ and $(c\tau + d)^2$ behave well w.r.t. matrix multiplication.

8. Set $\tilde{G}_2(\tau) = -\dfrac{1}{8\pi^2} G_2(\tau)$. Show that

(a) $\tilde{G}_2(\tau) = \dfrac{1}{2}\zeta(-1) + \displaystyle\sum_{n=1}^{\infty} \sigma_1(n) q^n$

(b) $\tilde{G}_2\left(\dfrac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^2 \tilde{G}_2(\tau) - \dfrac{1}{4\pi i} c(c\tau + d)$, for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z})$ and $\tau \in \mathcal{H}$.