HOMEWORK 5

DUE 21 OCTOBER 2008

This Pset outlines the proof of one of the most beautiful and useful formulas in mathematics, the *Jacobi triple product formula*. It has connections with representation theory, number theory, as well as physics, combinatorics and computational complexity. It states that for complex numbers $0 < |q| < 1$ and $x \neq 0$

$$\sum_{n=-\infty}^{\infty} q^{n^2} x^n = \prod_{n=1}^{\infty} (1 - q^{2n})(1 + q^{2n-1}x)(1 + q^{2n-1}x^{-1})$$

Consider parameters $z \in \mathcal{H}$ and $w \in \mathbb{C}$. Denote $\Lambda(z) = \{2mz + n; m, n \in \mathbb{Z}\}$ the lattice spanned by $2z$ and 1. We'll write $q = e^{2\pi i z}$ and $x = e^{2\pi i w}$. As z and w vary in their respective domains, q and x describe the sets mentioned in the formula.

Definition An *elliptic function* with respect to a lattice $\Gamma \subset \mathbb{C}$ is a meromorphic function f on $\mathbb{C}$ such that $f(u + \gamma) = f(u)$ for any $u \in \mathbb{C}$ and $\gamma \in \Gamma$.

1. Use the maximum modulus principle to show that if a f is elliptic with respect to some lattice Γ, but has no poles, then it is constant.

2. Define $\vartheta(z, w) = \sum_{n=-\infty}^{\infty} q^{n^2} x^n$ and $P(z, w) = \prod_{n=1}^{\infty}(1 + q^{2n-1}x)(1 + q^{2n-1}x^{-1})$. Show that

$$\vartheta(z, w + 2z) = (qx)^{-1}\vartheta(z, w) \text{ and } P(z, w + 2z) = (qx)^{-1}P(z, w).$$

Deduce that, for fixed z, the function $f_z(w) = \dfrac{\vartheta(z, w)}{P(z, w)}$ is elliptic with respect to the lattice $\Lambda(z)$.

3. Fix z. Prove that if $P(z, w) = 0$, then either $w = \frac{1}{2} + z + \lambda$ or $w = \frac{1}{2} - z + \lambda$ for some $\lambda \in \Lambda(z)$ and those zeroes are necessarily simple. Show that these values are also zeros of $\vartheta(z, w)$ and conclude that $f_z(w)$ has no poles.

Deduce that for any z and w,

$$\vartheta(z, w) = \phi(q)P(z, w)$$

for some function $\phi(q)$ independent of w .

4. The Jacobi triple product formula will follow if we show that

$$\phi(q) = \prod_{n=1}^{\infty}(1 - q^{2n}). \tag{1}$$

To this end, show that

$$\vartheta\left(4z, \frac{1}{2}\right) = \vartheta\left(z, \frac{1}{4}\right)$$

and

$$\frac{P\left(4z, \frac{1}{2}\right)}{P\left(z, \frac{1}{4}\right)} = \prod_{n=1}^{\infty}(1 - q^{4n-2})(1 - q^{8n-4}).$$

Also show that

$$\phi(q) = \frac{P\left(4z, \frac{1}{2}\right)}{P\left(z, \frac{1}{4}\right)}\,\phi(q^4).$$

Now show that $\lim_{q\to 0} \phi(q) = 1$ and deduce (1).