HOMEWORK 6

DUE 28 OCTOBER 2008

1. This example shows that the condition of moderate growth in the Phragmén-Lindelöf principle is necessary. Let

$$f(s) = e^{e^{-is}}.$$

Show that f is bounded on the lines $\Re(s) = \pm\frac{\pi}{2}$, but it is not bounded in the strip $-\frac{\pi}{2} < \Re(s) < \frac{\pi}{2}$.

2. Show that the measure on $\mathcal{H}$ given by $\dfrac{dxdy}{y^2}$ is invariant under the action of the modular group G. Compute the volume of $G\backslash\mathcal{H}$ with respect to this measure.

3. The Peterson inner product on the space of cusp forms of weight $2k$ is given by

$$\langle f, g\rangle = \int_{G\backslash\mathcal{H}} f(z)\overline{g(z)}y^{2k}\frac{dxdy}{y^2}.$$

Show that $\langle T(n)f, g\rangle = \langle f, T(n)g\rangle$ for any integer $n \geq 1$ and any $f, g \in M_k^\circ$, i.e. $T(n)$ is self-adjoint with respect to the Peterson inner product.

4. The function $\eta : \mathcal{H} \to \mathbb{C}$, $\eta(z) = q^{1/24}\displaystyle\prod_{n=1}^{\infty}(1-q^n)$ is known as the *Dedekind eta function.* Use the Jacobi triple product formula to prove that

$$\eta(z) = \sum_{n=1}^{\infty}\chi(n)q^{n^2/24}, \text{ where } \chi(n) = \begin{cases} 1 & \text{if } n \equiv \pm 1 \pmod{12} \\ -1 & \text{if } n \equiv \pm 5 \pmod{12} \\ 0 & \text{otherwise.}\end{cases}$$

Show that χ is the primitive quadratic character mod 12.

5. Show that if $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}(2,\mathbb{Z})$ and any $z \in \mathcal{H}$ there exists a 24th root of unity $\epsilon(g)$ such that

$$\eta\left(\frac{az+b}{cz+d}\right) = \epsilon(g)(cz+d)^{1/2}\eta(z).$$

Note: There is an ambiguity about the sign in the choice of the square root $(cz+d)^{1/2}$. But, because we are only asserting that $\epsilon(g)$ lies in the group of 24th roots of unity, this is not a problem. Also note that this implies (again!) the modular property for Ramanujan's Δ.