HOMEWORK 7 – PART II

DUE 11 NOVEMBER 2008

Reading: chapter I in Koblitz's book and chapter II, section 1 in Serre's book.

1. Let F be a field endowed with a non-archimedean norm $\| \ \|$. The open ball of radius $r \in (0, \infty)$ centered at $a \in F$ is, as usual, $B(a, r) = \{x \in F; \|x - a\| < r\}$. Prove that any element of this radius is also a center, namely that for any $x \in B(a, r)$ we have $B(a, r) = B(x, r)$. Prove the same thing for the closed ball.

2. Prove that if $a \in \mathbb{Q}$, $a \neq 0$ and $|a|_p \leq 1$ for all primes p, then a is an integer.

3. Prove that $v_p((p^n)!) = 1 + p + \cdots + p^{n-1}$ and that $v_p((ap^n)!) = a(1 + p + \cdots + p^{n-1})$ if $0 \leq a \leq p - 1$.

4. Prove that of x is a nonzero rational number, then $\prod_p |x|_p = 1$ where the product is taken over all primes p and ∞.

5. If $a \in \mathbb{Q}_p$ has p-adic expansion $a = \sum_{n \geq -m} a_n p^n$, what is the p-adic expansion of $-a$?

6. Prove that the p-adic expansion of $a \in \mathbb{Q}_p$ has finitely many non-zero terms if and only if a is a positive rational number whose denominator is a power of p.

7. Prove that the p-adic expansion of $a \in \mathbb{Q}_p$ has repeating digits from some point on (i.e., $a_{i+r} = a_i$ for some r and for all i greater than some N) if and only if $a \in \mathbb{Q}$.

8. Compute the first 6 digits of the p-adic expansion of $\pm\sqrt{-1}$ in $\mathbb{Q}_5$ and $\pm\sqrt{-3}$ in $\mathbb{Q}_7$.

9. What is the cardinality of $\mathbb{Z}_p$? Prove your answer.

10. Find the p-adic expansion of:
(a) 75 in $\mathbb{Q}_2$

(b) 2/3 in $\mathbb{Q}_2$

(c) $-1/1000$ in $\mathbb{Q}_5$

(d) $-9/16$ in $\mathbb{Q}_7$

(e) 2/3 in $\mathbb{Q}_3$

(f) $\pm 1/6$ in $\mathbb{Q}_7$ and $\mathbb{Q}_{13}$

11. Which of the following 11-adic numbers have square roots in $\mathbb{Q}_{11}$? Justify your answer. If the square roots exist, find the first 3 digits in their 11-adic expansion.

(a) ± 7

(b) $1 + 3 \cdot 11^3$

(c) $3 \cdot 11^{-2} + 6 \cdot 11^{-1} + 3 + 7 \cdot 11^2$

(d) $1 \cdot 11^7$

(e) $7 - 6 \cdot 11^2$