Circle valued momentum maps for symplectic periodic flows

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Abstract

We give a detailed proof of the well-known classical fact that every symplectic circle action on a compact manifold admits a circle valued momentum map relative to some symplectic form. This momentum map is Morse-Bott-Novikov and each connected component of the fixed point set has even index. These proofs do not appear to be written elsewhere.

1 Introduction

A smooth circle action on a symplectic manifold (M, σ) is *Hamiltonian* if it is symplectic (in other words, the diffeomorphism associated to every group element preserves the symplectic form) and the contraction $\mathbf{i}_{1_M}\sigma := \sigma(1_M, \cdot)$ of σ with the infinitesimal generator 1_M of the action is an exact one form, i.e., it is of the form $\mathbf{d}\mu$ for some smooth function $\mu\colon M \to \mathbb{R}$. The map μ is called a *momentum map* of the action. Any two momentum maps differ by a constant on each connected component of M.

While there are many examples of interesting Hamiltonian circle actions – see for example Karshon's classification [9] in dimension 4 – there are also numerous situations in geometry and dynamical systems when one has a symplectic circle action (equivalently, a symplectic periodic flow) on a manifold but the one-form $\mathbf{i}_{\xi_M}\sigma$ is not exact, e.g., consider any action without fixed points such as a free action. Duistermaat-Pelayo [3, Remark 7.6 and Theorem 9.6] and Pelayo [19, Examples 8.1.1, 8.1.2, and Theorem 8.2.1] give infinitely many examples of compact connected symplectic manifolds in any dimension equipped with symplectic free torus actions that are hence not Hamiltonian. A particularly famous example among these manifolds is the *Kodaira variety* in [10, Theorem 19, case 3], also known as the *Kodaira-Thurston manifold* [13, Example 3.8 on page 88], which was pointed out by Thurston [21] to be a non-Kähler symplectic manifold.

The simplest example of a Hamiltonian S^1-action is the rotation of the sphere S^2 about the polar axis. The flow lines of the infinitesimal generator defining this action are the latitude circles. Frankel's seminal results [5, Lemmas 1 and 2] and their proofs imply that the momentum map for a circle action on a compact Kähler manifold is Morse-Bott and the index of each connected component of the fixed point set of the action is even.

The goal of this expository note is to give a proof of two classical facts:

(a) *Every symplectic circle action admits a circle valued momentum map relative to some (possibly different) invariant symplectic form.*

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Concretely, (a) means the following. Consider a symplectic S^1-action on a compact connected symplectic manifold M, with generating vector field 1_M. We are identifying the circle S^1 with the quotient $\mathbb{R}/\mathbb{Z}$ having coordinate t and length form $\lambda = dt$. Then $\mathbf{i}_{1_M}\sigma$ is a closed one-form and so it represents a cohomology class in $H^1(M;\mathbb{R})$. If this class is in the image of $H^1(M;\mathbb{Z}) \to H^1(M;\mathbb{R})$, then the action has a "circle valued momentum map" $\mu\colon M \to S^1$ with the property that $\mathbf{i}_{1_M}\sigma = \mu^*(dt)$. If not, then there is a (nearby) S^1-invariant symplectic form ω such that $k[\mathbf{i}_{1_M}\omega] \in H^1(M;\mathbb{Z})$ for some $k \in \mathbb{R}$. There is always a corresponding momentum[1] map $\mu\colon M \to S^1$ for any such form $k\omega$. This important observation was first made in an influential paper by Dusa McDuff [12, Lemma 1], which prompted much later research.

(b) *The map $\mu\colon M \to S^1$ is Morse-Bott-Novikov and each connected component of the fixed point set has even index.*

Statement (b) extends Frankel's result to circle valued momentum maps. Strictly speaking, in order to state this latter fact precisely we need to first introduce the notions of "Morse-Bott-Novikov", "non-degeneracy", and "index" of a critical point for smooth circle valued maps. The definitions of these notions parallel those for real valued maps. Formally, one replaces the smooth S^1-valued function by its logarithmic exterior differential which is in agreement with the fact that "non-degeneracy" of a critical point is a local notion. Once we have introduced these concepts, we can state the main result of this note, Theorem 3 in Section 2, stating McDuff's important observation and the extension of Frankel's theorem.

To our knowledge, no detailed proofs of these classical facts are available in the literature. We have received many questions over the years about them. This has prompted us to write the present note, with the goal to provide elementary and complete proofs. We follow, to a certain extent, McDuff's outline in the proof of the existence of the circle valued momentum map and extend Frankel's argument from real to circle valued maps. The proofs in this note are self-contained.

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2 Circle valued maps and the main theorem

In this section we explain in which sense a circle valued momentum map is "Morse-Bott-Novikov", what non-degeneracy means for a critical point of such a map, and what is the natural notion of index of a critical point. To be as self-contained as possible, we start by recalling some basic notions.

Morse theory for circle-valued momentum maps

Throughout this subsection M is any smooth manifold (not necessarily connected, or compact, or symplectic, etc.)

Conventions concerning S^1. Our conventions and notations concerning the circle S^1 are the following. We identify throughout this paper the circle S^1 with $\mathbb{R}/\mathbb{Z}$ and denote by $\pi : \mathbb{R} \ni t \mapsto [t] \in \mathbb{R}/\mathbb{Z}$ the canonical projection, a surjective submersive Lie group homomorphism. Thus $T_0\pi : \mathbb{R} \to T_{[0]}(\mathbb{R}/\mathbb{Z})$ is an isomorphism and so we identify $\mathbb{R}$ with the Lie algebra of $\mathbb{R}/\mathbb{Z}$, i.e., $r \in \mathbb{R}$ is identified with $T_0\pi(r)$. If

[1] If t is the coordinate on $\mathbb{R}$ and $\mu\colon M \to \mathbb{R}$ is the standard momentum map then $\mathbf{d}\mu = \mu^*(dt)$; formally, this information is found in formula (2) in Section 2.

L_t and $L_{[t]}$ denote left (equivalently, right) translation on $\mathbb{R}$ and $\mathbb{R}/\mathbb{Z}$, respectively, then $\pi \circ L_t = L_{[t]} \circ \pi$ and hence

$$T_{[t]}(\mathbb{R}/\mathbb{Z}) = \{T_t\pi(r) \mid r \in \mathbb{R}\}, \qquad \left(T_{[0]}L_{[t]} \circ T_0\pi\right)(0, r) = T_t\pi(t, r), \ \forall t, r \in \mathbb{R}.$$

The length form $\lambda \in \Omega^1(\mathbb{R}/\mathbb{Z})$ is defined by $\lambda([t])\,(T_t\pi(r)) := r$. So, in the local coordinate $t \in I$ ($I \subset \mathbb{R}$ is an open interval of length strictly less than one), $\lambda = \mathbf{d}t$ since $T_t\pi(r) = r\frac{\partial}{\partial t}$. Therefore, $\int_{\mathbb{R}/\mathbb{Z}} \lambda = \int_0^1 \mathbf{d}t = 1$ and λ is left (equivalently, right) invariant.

Logarithmic exterior differential. For any smooth map $f : M \to \mathbb{R}/\mathbb{Z}$, the classical *logarithmic exterior differential* $\boldsymbol{\delta} f \in \Omega^1(M)$ of f is defined by

$$\boldsymbol{\delta} f(m)(v_m) := T_{f(m)}L_{-f(m)}\left(T_m f(v_m)\right) \in \mathbb{R}, \tag{1}$$

where $m \in M$, $v_m \in T_mM$. It is easy to see that if $g : M \to \mathbb{R}/\mathbb{Z}$ is another smooth map, then $\boldsymbol{\delta}(fg) = \boldsymbol{\delta} f + \boldsymbol{\delta} g$. As usual, for $X \in \mathfrak{X}(M)$, we define $\langle \boldsymbol{\delta} f, X\rangle \in \mathrm{C}^\infty(M)$ by $\langle \boldsymbol{\delta} f, X\rangle(m) := \boldsymbol{\delta} f(m)(X(m))$ for any $m \in M$.

The following formula for any $f : M \to \mathbb{R}/\mathbb{Z}$ is easy to check and will be used later on:

$$f^*\lambda = \boldsymbol{\delta} f. \tag{2}$$

The logarithmic exterior differential is related to the usual exterior differential of the canonical lift in the following manner. Let $\widetilde{M} := \{(m, t) \in M \times \mathbb{R} \mid f(m) = [t]\}$ be the pull back bundle by f of the principal $\mathbb{Z}$-bundle $\pi : \mathbb{R} \to \mathbb{R}/\mathbb{Z}$. Thus, $\widetilde{\pi} : \widetilde{M} \ni (m, t) \mapsto m \in M$ is also a principal $\mathbb{Z}$-bundle and hence a covering space. Define the *canonical lift* of f by $\tilde{f} : \widetilde{M} \ni (m, t) \mapsto t \in \mathbb{R}$; thus we have $\pi \circ \tilde{f} = f \circ \widetilde{\pi}$ which implies that

$$\boldsymbol{\delta} f(m)(v_m) = \mathbf{d}\tilde{f}(m, t)\,(v_m, (t, r)) \tag{3}$$

for all $m \in M$, $v_m \in T_mM$, $t, r \in \mathbb{R}$. In particular, $m \in M$ *is a critical point of* f (i.e., $T_mf = 0$ or, equivalently, $\boldsymbol{\delta} f(m) = 0$) *if and only if all* $(m, t) \in \widetilde{\pi}^{-1}(m) \subset \widetilde{M}$ *are critical points of the real valued function* $\tilde{f}$. Denote by $\mathrm{Crit}(f) := \{m \in M \mid \boldsymbol{\delta} f(m) = 0\}$ the set of critical points of f.

Hessian of a circle-valued smooth map. The definition of the Hessian $(\operatorname{Hess} f)(m_0) : T_{m_0}M \times T_{m_0}M \to \mathbb{R}$ at the critical point $m_0 \in M$ of $f : M \to \mathbb{R}/\mathbb{Z}$ parallels that for real valued functions (see, e.g., [14, page 4]):

$$(\operatorname{Hess} f)(m_0)(u, v) := \boldsymbol{\pounds}_{\widetilde{u}}\left(\langle \boldsymbol{\delta} f, \widetilde{v}\rangle\right)(m_0) = \left\langle \mathbf{d}\left(\langle \boldsymbol{\delta} f, \widetilde{v}\rangle\right)(m_0), u\right\rangle, \tag{4}$$

for all $u, v \in T_{m_0}M$, where $\widetilde{u}, \widetilde{v}$ are arbitrary local smooth vector fields in a neighborhood of m_0 such that $\widetilde{u}(m_0) = u$, $\widetilde{v}(m_0) = v$. From (3) it follows that if $m_0 \in \mathrm{Crit}(f)$, then

$$(\operatorname{Hess} f)(m_0)(u, v) = (\operatorname{Hess} \tilde{f})(m_0, t_0)\big((u, (t_0, r)), (v, (t_0, s))\big), \tag{5}$$

for any $t_0 \in \mathbb{R}$ satisfying $f(m_0) = [t_0]$, $u, v \in T_{m_0}M$, $r, s \in \mathbb{R}$, where the right hand side is the usual Hessian of the real-valued function $\tilde{f}$. To see this, note first that

$$T_{(m,t)}\widetilde{M} = \left\{(u, (t, r)) \mid u \in T_mM,\ t, r \in \mathbb{R},\ T_mf(u) = T_t\pi(t, r) = \left(T_{[0]}L_{[t]} \circ T_0\pi\right)(0, r)\right\}.$$

Thus, if $v \in T_{m_0}M$ and $\widetilde{v}$ is an arbitrary local smooth vector field defined in a neighborhood of m_0 and satisfying $\widetilde{v}(m_0) = v$, then $(\widetilde{v}, \tilde{s})$ is a smooth local vector field defined in a neighborhood of $(m_0, t_0) \in$

$\widetilde{M}$ whose value at (m_0, t_0) is $(v, (t_0, s)) \in T_{(m_0,t_0)}\widetilde{M}$, provided that $\tilde{s}(t) := T_{[t]}L_{-[t]}T_m f(\widetilde{v}(m)) \in \mathbb{R}$, for all m in the domain of definition of $\widetilde{v}$. Thus, if $m(\varepsilon) \in M$, $t(\varepsilon) = t_0 + \varepsilon r$, with $m(0) = m_0$ and $m'(0) = u$, we get

$$\begin{aligned}(\mathrm{Hess}\, \tilde{f})(m_0, t_0)\big((u, (t_0, r)), (v, (t_0, s))\big) &= \left\langle \mathbf{d} \left\langle \mathbf{d}\tilde{f}, (\widetilde{v}, \tilde{s}) \right\rangle (m_0, t_0), (u, (t_0, r)) \right\rangle \\ &= \left.\frac{d}{d\epsilon}\right|_{\epsilon=0} \left\langle \mathbf{d}\tilde{f}, (\widetilde{v}, \tilde{s}) \right\rangle (m(\varepsilon), t_0 + \varepsilon r) \overset{(3)}{=} \left.\frac{d}{d\epsilon}\right|_{\epsilon=0} \langle \boldsymbol{\delta} f, \widetilde{v} \rangle (m(\varepsilon)) \\ &= \langle \mathbf{d} (\langle \boldsymbol{\delta} f, \widetilde{v} \rangle) (m_0), u \rangle \overset{(4)}{=} (\mathrm{Hess}\, f)(m_0)(u, v).\end{aligned}$$

Formula (5) shows that $(\mathrm{Hess}\, f)(m_0) : T_{m_0}M \times T_{m_0}M \to \mathbb{R}$ is a symmetric bilinear form and that its definition does not depend on the extensions $\widetilde{u}$ and $\widetilde{v}$ but only on their point values $u, v \in T_{m_0}M$. As for real-valued functions, the critical point m_0 is said to be *non-degenerate* if $(\mathrm{Hess}\, f)(m_0)$ is a non-degenerate bilinear form. Thus, formula (5) implies that m_0 *is a non-degenerate critical point of f if and only if all $(m_0, t_0) \in \widetilde{\pi}^{-1}(m_0) \subset \widetilde{M}$ are non-degenerate critical points of $\tilde{f}$*. In addition, the Morse Lemma for $\tilde{f}$ and the fact that $\widetilde{\pi} : \widetilde{M} \to M$ is a covering space, implies that *non-degenerate critical points of $f : M \to \mathbb{R}/\mathbb{Z}$ are isolated*. In particular, if M is compact, then there are only finitely many non-degenerate critical points of f.

Morse-Bott-Novikov maps. Recall that a smooth map $f : M \to \mathbb{R}$ is *Morse* if all its critical points are non-degenerate. The smooth map f is *Morse-Bott* if the critical set $\mathrm{Crit}(f)$ of f is a disjoint union of connected submanifolds C_i of M such that $\ker(\mathrm{Hess}\, f)(m) = T_mC_i$, for each i and $m \in C_i$. The *index of* m is the number of negative eigenvalues of $(\mathrm{Hess}\, f)(m)$. For circle valued maps we proceed in the same manner.

A smooth map $f : M \to \mathbb{R}/\mathbb{Z}$ is *Morse-Bott-Novikov* if the critical set $\mathrm{Crit}(f) := \{m \in M \mid \boldsymbol{\delta} f(m) = 0\}$ of f is a disjoint union of connected submanifolds C_i of M such that $\ker(\mathrm{Hess}\, f)(m) = T_mC_i$, for each i and $m \in C_i$. The *index of* m is the number of negative eigenvalues of $(\mathrm{Hess}\, f)(m)$. Since $\mathrm{Crit}(f)$ is closed, if M is compact, then it has only a finite number of connected components.

Formula (3) implies that $\mathrm{Crit}\, \tilde{f} = \widetilde{\pi}^{-1}(\mathrm{Crit}\, f)$. Thus, from (5) we conclude that $f : M \to \mathbb{R}/\mathbb{Z}$ *is Morse-Bott-Novikov if and only if $\tilde{f} : \widetilde{M} \to \mathbb{R}$ is Morse-Bott*.

The circle-valued momentum map

Let (M, σ) be a symplectic manifold, i.e., M is a smooth manifold and σ is a non-degenerate closed smooth 2-form on M.

Let $\Phi : \mathbb{R}/\mathbb{Z} \times M \to M$ be a smooth action by symplectomorphisms (i.e., each diffeomorphism $\Phi_{[t]} : M \to M$ preserves the symplectic form σ). Let $r_M \in \mathfrak{X}(M)$ be the infinitesimal generator of the action determined by $r \in \mathbb{R}$ whose value at an arbitrary point $x \in M$ is given by

$$r_M(x) := \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \Phi_{[r\varepsilon]}(x).$$

The circle action on (M, σ) is said to be *Hamiltonian* if there exists a smooth map $\mu \colon M \to \mathbb{R}$, called the *momentum map*, such that $\mathbf{i}_{1_M}\sigma = \sigma(1_M, \cdot) = \mathbf{d}\mu$. The existence of such a map μ is equivalent to the exactness of the one-form $\mathbf{i}_{1_M}\sigma$. It follows that the obstruction to the action being Hamiltonian lies in the first cohomology group of M; thus, if $H^1(M; \mathbb{R})$ is the trivial group then every symplectic $\mathbb{R}/\mathbb{Z}$-action on M is Hamiltonian.

Definition 1. *A* circle valued momentum map $\mu : M \to \mathbb{R}/\mathbb{Z}$ *is defined by the condition $\mu^*\lambda = \mathbf{i}_{1_M}\sigma$, where $\lambda \in \Omega^1(\mathbb{R}/\mathbb{Z})$ the standard length form.*

Remark 2. Definition 1 is equivalent to that of group valued momentum maps ([1, 7, 8], [17, Definition 5.4.1]) in the case of $\mathbb{R}/\mathbb{Z}$ because of formula (2). There is a close relationship between the cylinder valued momentum map ([2], [17, §5.2]) and the group valued momentum map for Abelian Lie groups. Any cylinder valued momentum map associated to an Abelian Lie algebra action whose associated holonomy group is closed can be understood as a Lie group valued momentum map ([17, Proposition 5.4.4]). Conversely, connected Abelian Lie groups have closed holonomy groups. The precise technical conditions when Lie group and cylinder valued momentum maps are equivalent for connected Abelian Lie groups are spelled out in [17, Theorem 5.4.6]. ⊘

Statement of the main theorem. With this background, we can now give a precise statement of the result we announced in the introduction.

Theorem 3. *Let the circle $\mathbb{R}/\mathbb{Z}$ act symplectically on the compact symplectic manifold (M, σ). Denote by $M^{\mathbb{R}/\mathbb{Z}}$ the fixed point set of the $\mathbb{R}/\mathbb{Z}$-action. Then either the action admits a standard momentum map or, if not, there exists a $\mathbb{R}/\mathbb{Z}$-invariant symplectic form ω on M that admits a circle valued momentum map $\mu : M \to \mathbb{R}/\mathbb{Z}$. Moreover, μ is a Morse-Bott-Novikov function and each connected component of $M^{\mathbb{R}/\mathbb{Z}} = \mathrm{Crit}(\mu)$ has even index.*

Remark 4. As we shall see in the proof, if σ is integral, then $\omega = \sigma$. ⊘

Remark 5. An analogue of Theorem 3 also holds for actions of higher dimensional tori. ⊘

Remark 6. If $\mu : M \to \mathbb{R}$ is a standard momentum map for a circle action on a $2n$-dimensional compact symplectic manifold (M, σ), it is well-known that it has at least $n+1$ critical points or, equivalently, the circle action has at least $n+1$ fixed points. Let us briefly recall the argument. Since μ is Morse-Bott (Theorem 3), the connected components of $\mathrm{Crit}(\mu)$ are submanifolds of M. If at least one is not zero dimensional, then there are infinitely many critical points of μ and the result is obvious. If all connected components are zero dimensional, then μ is a Morse function and so it must be perfect (i.e., the Morse inequalities are equalities) because of the following classical result: *If f is a Morse function on a compact manifold whose critical points have only even indices, then it is a perfect Morse function* (see e.g., [15, Corollary 2.19 on page 52]). Thus, if $m_k(\mu)$ denotes the number of critical points of μ of index k, the total number of critical points of μ equals $\sum_{k=0}^{2n} m_k(\mu) = \sum_{k=0}^{2n} \mathrm{b}_k(M)$, where $\mathrm{b}_k(M) := \dim\left(H^k(M, \mathbb{R})\right)$ is the kth Betti number of M. However, since σ is a symplectic form, the cohomology classes $[\sigma^k]$ are nontrivial elements of $H^{2k}(M, \mathbb{R})$ for $k = 0, \ldots, n$, and hence $\mathrm{b}_{2k}(M) \geq 1$, which then implies that the total number of critical points of μ is at least $n+1$.

It is tempting to use Theorem 3 to deduce a similar result for circle valued momentum maps by replacing the Morse inequalities by the Novikov inequalities (see [18, Chapter 11, Proposition 2.4], [4, Theorem 2.4]), if all critical points of μ are non-degenerate. In this case, the number of critical points of the circle-valued momentum map μ is $\sum_{k=0}^{2n} m_k(\mu)$. This integer is estimated from below by

$$\sum_{k=0}^{2n} \Big(\hat{b}_k(M) + \hat{q}_k(M) + \hat{q}_{k-1}(M) \Big),$$

where $\hat{b}_k(M)$ is the rank of the $\mathbb{Z}((t))$-module $H_k(\widetilde{M}, \mathbb{Z}) \otimes_{\mathbb{Z}[t,t^{-1}]} \mathbb{Z}((t))$, $\hat{q}_k(M)$ is the torsion number of this module, and $\widetilde{M}$ is the pull back by $\mu : M \to \mathbb{R}/\mathbb{Z}$ of the principal $\mathbb{Z}$-bundle $t \in \mathbb{R} \mapsto [t] \in \mathbb{R}/\mathbb{Z}$. Unfortunately, this lower bound can be zero, in stark contrast to the Hamiltonian case. For example, the circle action on the two-torus by rotation on the first factor is free and hence has no fixed points. See [4, §7.3] for further information. However, it is known that if this lower bound is strictly positive, then it must be at least two. In addition, if $\dim(M) \geq 8$, then if the lower bound is strictly positive, it must be at least three. These results were proved in [20, Corollary 6] using localization in equivariant cohomology. To our knowledge, no universal lower bound for non-Hamiltonian symplectic circle actions with at least one fixed point is available. It was proved in [20, Theorem 1] that this lower bound is at least $n+1$ provided that the so-called Chern class map is somewhere injective. $\oslash$

The rest of the paper is devoted to the proof of Theorem 3.

3 Proof of the first part of Theorem 3: existence of μ

The goal of this section is to prove the existence of the circle valued momentum map.

Notation and basic facts

In order to be as explicit and self-contained as possible we give a proof of the following basic observation.

Lemma 7. *Let $\Phi : (\mathbb{R}/\mathbb{Z}) \times M \to M$ be a smooth action and let $\varphi \colon \mathbb{R}/\mathbb{Z} \to N$ be a smooth map. Define $\psi : (\mathbb{R}/\mathbb{Z}) \times (\mathbb{R}/\mathbb{Z}) \to N$ by*

$$\psi([s],[t]) := \Phi_{[t]}\left(\varphi\left([s]\right)\right).$$

Then, if $\alpha \in \Omega^2(M)$ is an $\mathbb{R}/\mathbb{Z}$-invariant form we have

$$\int_{(\mathbb{R}/\mathbb{Z})\times(\mathbb{R}/\mathbb{Z})} \psi^*\alpha = -\int_{\mathbb{R}/\mathbb{Z}} \varphi^*(\mathbf{i}_{1_M}\alpha). \tag{6}$$

Proof. Let $\beta \in \Omega^2((\mathbb{R}/\mathbb{Z}) \times (\mathbb{R}/\mathbb{Z}))$. Denote by $[s]$ the elements of the first circle and by $[t]$ those of the second. Let $\partial/\partial t \in \mathfrak{X}(\mathbb{R}/\mathbb{Z})$ be the left (equivalently, right) invariant vector field whose value at $[0]$ is 1. Let $\mathbf{d}t \in \Omega^1(\mathbb{R}/\mathbb{Z})$ be the one-form dual to $\partial/\partial t$, i.e., $\langle \mathbf{d}t, \partial/\partial t\rangle = 1$. A direct verification shows that $\beta = -\mathbf{i}_{\frac{\partial}{\partial t}}\beta \wedge \mathbf{d}t$. If β is invariant under the translations $\Lambda_{[u]}$ of the second factor in $(\mathbb{R}/\mathbb{Z}) \times (\mathbb{R}/\mathbb{Z})$ (the t-direction), i.e., if $\Lambda_{[u]}([s],[t]) := ([s],[t+u])$, then $\Lambda_{[u]}^*\beta = \beta$ for all $[u] \in \mathbb{R}/\mathbb{Z}$, it follows that $\mathbf{i}_{\frac{\partial}{\partial t}}\beta$ is also invariant under such translations. Thus $\mathbf{i}_{\frac{\partial}{\partial t}}\beta$ depends only on $[s]$ and hence

$$\int_{(\mathbb{R}/\mathbb{Z})\times(\mathbb{R}/\mathbb{Z})} \beta = -\int_{\mathbb{R}/\mathbb{Z}} \iota_1^*\mathbf{i}_{\frac{\partial}{\partial t}}\beta, \tag{7}$$

where $\iota_1 : (\mathbb{R}/\mathbb{Z}) \ni [s] \longmapsto ([s],[0]) \in (\mathbb{R}/\mathbb{Z}) \times (\mathbb{R}/\mathbb{Z})$ is the standard embedding of the first circle into the 2-torus.

Now notice that $\psi^*\alpha \in \Omega^2((\mathbb{R}/\mathbb{Z}) \times (\mathbb{R}/\mathbb{Z}))$ is invariant under the translations on the second factor. Indeed, since

$$\begin{aligned} T_{([s],[t])}\psi\left(a\frac{\partial}{\partial s}, b\frac{\partial}{\partial t}\right) &= T_{\varphi([s])}\Phi_{[t]}\left(aT_{[s]}\varphi\left(\frac{\partial}{\partial s}\right) + b1_M(\varphi([s]))\right) \\ T_{([s],[t])}\Lambda_{[u]}\left(a\frac{\partial}{\partial s}, b\frac{\partial}{\partial t}\right) &= \left(a\frac{\partial}{\partial s}, b\frac{\partial}{\partial t}\right)([s],[t+u]), \end{aligned}$$

$(\mathbb{R}/\mathbb{Z})$-invariance of $\alpha \in \Omega^2(M)$ implies that $\Lambda_{[u]}^*\psi^*\alpha = \psi^*\alpha$. Thus, formula (7) is applicable for

$\beta = \psi^*\alpha$. In addition,

$$\left(\iota_1^* \mathbf{i}_{\frac{\partial}{\partial t}} \psi^*\alpha\right)([s])\left(a\frac{\partial}{\partial s}\right) = \left(\mathbf{i}_{\frac{\partial}{\partial t}} \psi^*\alpha\right)([s],[0])\left(a\frac{\partial}{\partial s},0\right) = (\psi^*\alpha)([s],[0])\left(\left(0,\frac{\partial}{\partial t}\right),\left(a\frac{\partial}{\partial s},0\right)\right)$$
$$= \alpha\left(\psi([s],[0])\right)\left(T_{([s],[0])}\psi\left(0,\frac{\partial}{\partial t}\right), T_{([s],[0])}\psi\left(a\frac{\partial}{\partial s},0\right)\right)$$
$$= \alpha\left(\varphi([s])\right)\left(1_M(\varphi([s])), aT_{[s]}\varphi\left(\frac{\partial}{\partial s}\right)\right)$$
$$= \left(\mathbf{i}_{1_M}\alpha\right)\left(\varphi([s])\right)\left(T_{[s]}\varphi\left(a\frac{\partial}{\partial s}\right)\right) = \varphi^*\left(\mathbf{i}_{1_M}\alpha\right)([s])\left(a\frac{\partial}{\partial s}\right),$$

i.e., $\iota_1^* \mathbf{i}_{\frac{\partial}{\partial t}} \psi^*\alpha = \varphi^*\left(\mathbf{i}_{1_M}\alpha\right)$ which, together with (7), implies formula (6). □

Existence of μ

If the $\mathbb{R}/\mathbb{Z}$-action does not admit a standard momentum map, the action is necessarily not trivial, because the trivial action admits the constant map everywhere equal to zero as a momentum map. Thus, assuming that the action is not Hamiltonian, it follows that the one-form $\mathbf{i}_{1_M}\sigma$ is not exact. In this case we shall prove that there exists a $\mathbb{R}/\mathbb{Z}$-invariant symplectic form ω on M that admits a circle valued momentum map $\mu : M \to \mathbb{R}/\mathbb{Z}$.

The following steps cover three cases. Before we proceed, we recall that for a compact manifold X, a *rational cohomology class* in $H^k(X;\mathbb{R})$ is a real cohomology class which lies in the image of $H^k(X;\mathbb{Q}) \to H^k(X;\mathbb{R})$. Similarly, when $\mathbb{Q}$ is replaced by $\mathbb{Z}$ for integral cohomology class.

Step 1. *Existence of the circle valued momentum map when* $[\sigma] \in H^2(M;\mathbb{Z})$. Identity (6) shows that $[\mathbf{i}_{1_M}\sigma] \in H^1(M;\mathbb{Z})$; we note that this statement may also be deduced as a property of the *flux homomorphism*, cf. [13, Lemma 10.7]. Pick a point $m_0 \in M$, let γ_m be an arbitrary smooth path connecting m_0 to m in M, and define the map $\mu : M \to \mathbb{R}/\mathbb{Z}$ by

$$\mu(m) := \left[\int_{\gamma_m} \mathbf{i}_{1_M}\sigma\right]. \tag{8}$$

The map μ is well defined. Indeed, if $\widetilde{\gamma}_m$ is another path connecting m_0 to m, let $\gamma_m * (-\widetilde{\gamma}_m)$ be the closed loop formed by starting at m_0, following γ_m and then returning from m to m_0 on $\widetilde{\gamma}_m$. Since $\mathbf{i}_{1_M}\sigma \in H^1(M;\mathbb{Z})$, all its periods are integral and hence $\int_{\gamma_m * (-\widetilde{\gamma}_m)} \mathbf{i}_{1_M}\sigma =: k \in \mathbb{Z}$. Thus

$$\int_{\gamma_m} \mathbf{i}_{1_M}\sigma = \int_{\widetilde{\gamma}_m} \mathbf{i}_{1_M}\sigma + k$$

which shows that $\left[\int_{\gamma_m} \mathbf{i}_{1_M}\sigma\right] = \left[\int_{\widetilde{\gamma}_m} \mathbf{i}_{1_M}\sigma\right]$. The map μ is clearly smooth. Finally, since for any $v_m \in T_mM$, we have $T_m\mu(v_m) = T_{\int_{\gamma_m} \mathbf{i}_{1_M}\sigma}\pi\big(\mathbf{i}_{1_M}\sigma(m)(v_m)\big)$, it follows that

$$(\mu^*\lambda)(m)(v_m) = \lambda(\mu(m))\left(T_m\mu(v_m)\right) = \left(\mathbf{i}_{1_M}\sigma\right)(m)(v_m),$$

i.e., the symplectic form σ admits the circle valued momentum map μ defined in (8) on M.

Step 2. *Existence of the circle valued momentum map when* $[\sigma] \in H^2(M;\mathbb{Q})$. Identity (6) shows that $[\mathbf{i}_{1_M}\sigma] \in H^1(M;\mathbb{Q})$. Thus there is a $k \in \mathbb{N}$ such that $[\mathbf{i}_{1_M}(k\sigma)] = k\,[\mathbf{i}_{1_M}\sigma] \in H^1(M;\mathbb{Z})$. Since the $\mathbb{R}/\mathbb{Z}$-action clearly preserves $k\sigma$, by Step 1, the symplectic form $k\sigma$ on M admits a circle valued momentum map on M.

Step 3. *Existence of the circle valued momentum map when* $[\sigma] \in H^2(M;\mathbb{R})$ *is irrational.* We will use the de Rham theorem for G-invariant forms: *let G be a connected compact Lie group acting smoothly on a compact manifold X. Let $\Omega^*(X)^G$ denote the set of G-invariant forms. Then the inclusion map $i\colon \Omega^*(X)^G \to \Omega^*(X)$ induces an isomorphism $H^*(X;\mathbb{R})^G \cong H^*(X;\mathbb{R})$ in real cohomology.*

Therefore, in our case, for the group $\mathbb{R}/\mathbb{Z}$, we conclude that $H^2(M;\mathbb{R})^{\mathbb{R}/\mathbb{Z}} \cong H^2(M;\mathbb{R})$ by the compactness of M; let $m := \dim_{\mathbb{R}} \left(H^2(M;\mathbb{R})\right) = \dim_{\mathbb{Q}} \left(H^2(M;\mathbb{Q})\right)$ be the second Betti number. Choose a $\mathbb{Q}$-basis of $H^2(M;\mathbb{Q})$; then it is also a $\mathbb{R}$-basis of $H^2(M;\mathbb{R}) \cong H^2(M;\mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{R}$ and hence $H^2(M;\mathbb{Q}) \cong \mathbb{Q}^m$ as $\mathbb{Q}$-vector spaces, $H^2(M;\mathbb{R}) \cong \mathbb{R}^m$ as $\mathbb{R}$-vector spaces. Endowing $H^2(M;\mathbb{R})$ with the topology induced by this linear isomorphism, this implies that $H^2(M;\mathbb{Q})$ is dense in $H^2(M;\mathbb{R}) \cong H^2(M;\mathbb{R})^{\mathbb{R}/\mathbb{Z}}$. Since $0 \neq [\sigma] \in H^2(M;\mathbb{R})^{\mathbb{R}/\mathbb{Z}}$ because σ is a symplectic form, we can complete to a basis $\{[\sigma], [\omega_1], \dots [\omega_{m-1}]\}$ of $H^2(M;\mathbb{R})^{\mathbb{R}/\mathbb{Z}}$. In particular, $\sigma, \omega_1, \dots, \omega_{m-1} \in \Omega^2_{\text{closed}}(M)^{\mathbb{R}/\mathbb{Z}}$ are linearly independent and hence $V := \operatorname{span}_{\mathbb{R}}\{\sigma, \omega_1, \dots \omega_{m-1}\}$ is a m-dimensional vector subspace of $\Omega^2_{\text{closed}}(M)^{\mathbb{R}/\mathbb{Z}}$ isomorphic to $H^2(M;\mathbb{R})^{\mathbb{R}/\mathbb{Z}}$, the isomorphism being given by its values on the basis: $\sigma \mapsto [\sigma]$, $\omega_k \mapsto [\omega_k]$, for $k = 1, \dots, m-1$. Embed by this isomorphism the $\mathbb{Q}$-vector space $H^2(M;\mathbb{Q})$ in V; its image U is a dense $\mathbb{Q}$-vector subspace of V. Because non-degeneracy is an open condition, it follows that the set of $\mathbb{R}/\mathbb{Z}$-invariant symplectic forms in V is *open* and also *non-empty* since $\sigma \in V$. Because U is dense in V, it follows that we can find a form $\omega \in U$, hence necessarily closed and $\mathbb{R}/\mathbb{Z}$-invariant, so close to $\sigma \in V$ that it is symplectic. The problem has now been reduced to the situation studied in Step 2 with σ replaced by ω.

This concludes the proof of existence of the circle valued momentum map.

4 Proof of the second part of Theorem 3: μ is Morse-Bott-Novikov

The goal of this section is to prove that the circle-valued momentum map $\mu : M \to \mathbb{R}/\mathbb{Z}$ is Morse-Bott-Novikov. As discussed in Section 2, this is equivalent to showing that the standard lift $\widetilde{\mu} : \widetilde{M} \to \mathbb{R}$ is Morse-Bott, where $\widetilde{M} = \{(m,t) \in M \times \mathbb{R} \mid \mu(m) = [t]\}$.

Let ω be the $\mathbb{R}/\mathbb{Z}$-invariant symplectic form on M constructed in Section 3. Since $\widetilde{\pi} : \widetilde{M} \ni (m,t) \mapsto m \in M$ is a covering space it follows that $\widetilde{\pi}^*\omega \in \Omega^2\left(\widetilde{M}\right)$ is a symplectic form on $\widetilde{M}$. In addition, $\mathbb{R}/\mathbb{Z}$ naturally acts on $\widetilde{M}$ by $\Psi_{[s]}(m,t) := (\Phi_{[s]}(m), t)$. This is well-defined since the momentum map is $(\mathbb{R}/\mathbb{Z})$-invariant. To see this, note that it suffices to prove that $T_m\mu(1_M(m)) = 0$, which follows from the following computation:

$$\begin{aligned} T_{\mu(m)}L_{-\mu(m)}T_m\mu(1_M(m)) &\overset{(1)}{=} \boldsymbol{\delta}\mu(m)(1_M(m)) \overset{(2)}{=} (\mu^*\lambda)(m)(1_M(m)) = (\mathbf{i}_{1_M}\omega)(m)(1_M(m)) \\ &= \omega(m)\left(1_M(m), 1_M(m)\right) = 0. \end{aligned}$$

The identity $\widetilde{\pi} \circ \Psi_{[s]} = \Phi_{[s]} \circ \widetilde{\pi}$ and $\mathbb{R}/\mathbb{Z}$-invariance of ω implies that the $\mathbb{R}/\mathbb{Z}$-action Ψ on $\widetilde{M}$ is symplectic. Let us show that $\widetilde{\mu} : \widetilde{M} \ni (m,t) \mapsto t \in \mathbb{R}$ is a momentum map of this action. Indeed, since $1_{\widetilde{M}}(m,t) = (1_M(m), (t,0))$, so $1_{\widetilde{M}}$ and 1_M are $\widetilde{\pi}$-related, for any $m \in M$, $v_m \in T_mM$, $t, r \in \mathbb{R}$, we have

$$\mathbf{i}_{1_{\widetilde{M}}}(\widetilde{\pi}^*\omega)(m,t)(v_m,(t,r)) = \mathbf{i}_{1_M}\omega(m)(v_m) = (\mu^*\lambda)(m)(v_m) \overset{(2)}{=} \boldsymbol{\delta}\mu(m)(v_m) \overset{(3)}{=} \mathbf{d}\widetilde{\mu}(m,t)(v_m,(t,r)).$$

Finally, note that $\widetilde{\mu} \circ \Psi_{[s]} = \widetilde{\mu}$. Thus, the problem is reduced to showing that the standard invariant momentum map of a circle action is Morse-Bott, which is a well-known classical result.

In the interest of completeness, we recall the proof. So, let (M,ω) be a compact symplectic manifold, $\Phi : (\mathbb{R}/\mathbb{Z}) \times M \to M$ an action preserving the symplectic form and admitting an invariant momentum map $\mathbf{J} : M \to \mathbb{R}$. We shall show that $\mathbf{J}$ is a Morse-Bott map.

First, we note that the submanifold $M^{\mathbb{R}/\mathbb{Z}}$, consisting of fixed points of the action, coincides with $\mathrm{Crit}(\mathbf{J})$. Indeed, since $\mathbb{R}/\mathbb{Z}$ is connected, so it is generated by a neighborhood of the identity element, it follows that $m \in M^{\mathbb{R}/\mathbb{Z}}$ if and only if $1_M(m) = 0$. By non-degeneracy of ω and the defining identity $\omega(m)(1_M(m), v_m) = \mathbf{dJ}(m)(v_m)$, for any $v_m \in T_m M$, of the momentum map $\mathbf{J}$, it follows that $1_M(m) = 0$ if and only if $\mathbf{dJ}(m) = 0$.

Second, we show that each connected component F of $M^{\mathbb{R}/\mathbb{Z}} = \mathrm{Crit}(\mathbf{J})$ has even index. Let $m_0 \in F \subset \mathrm{Crit}(\mathbf{J})$, $u, v \in T_{m_0}M$, and take vector fields $\widetilde{u}, \widetilde{v}$ such that $\widetilde{u}(m_0) = u$, $\widetilde{v}(m_0) = v$. Thus,

$$\begin{aligned}\mathrm{Hess}(\mathbf{J})(m_0) &= \boldsymbol{\pounds}_{\widetilde{u}}\left(\langle \mathbf{dJ}, \widetilde{v}\rangle\right)(m_0) = \boldsymbol{\pounds}_{\widetilde{u}}\left(\omega\left(1_M, \widetilde{v}\right)\right)(m_0)\\ &= \left(\boldsymbol{\pounds}_{\widetilde{u}}\omega\right)(m_0)\left(1_M(m_0), v\right) + \omega(m_0)\left(\left[\widetilde{u}, 1_M\right](m_0), v\right) + \omega(m_0)\left(1_M(m_0), \left[\widetilde{u}, \widetilde{v}\right](m_0)\right)\\ &= \omega(m_0)\left(v, \left[1_M, \widetilde{u}\right](m_0)\right) = \omega(m_0)\left(v, \left.\frac{d}{dt}\right|_{t=0}\left(T\Phi_{[-t]} \circ \widetilde{u} \circ \Phi_{[t]}\right)(m_0)\right)\\ &= \omega(m_0)\left(v, \left.\frac{d}{dt}\right|_{t=0} T_{m_0}\Phi_{[-t]}(u)\right).\end{aligned}$$

However, $T_{m_0}\Phi_{[t]} : T_{m_0}M \to T_{m_0}M$ is the flow of the linearized vector field $1'_M(m_0) : T_{m_0}M \to T_{m_0}M$ and hence

$$(\mathrm{Hess}\,\mathbf{J})(m_0)(u, v) = \omega(m_0)\left(v, -1'_M(m_0)(u)\right) = \omega(m_0)\left(1'_M(m_0)(u), v\right). \tag{9}$$

At this point we recall that the symplectic representation $T_{m_0}\Phi_{[t]} : (T_{m_0}M, \omega(m_0)) \to (T_{m_0}M, \omega(m_0))$ of $\mathbb{R}/\mathbb{Z}$ admits an invariant momentum map $\mathbf{L} : T_{m_0}M \to \mathbb{R}$ whose expression is

$$\mathbf{L}(v) = \frac{1}{2}\omega(m_0)\left(1'_M(m_0)(v), v\right)$$

for any $v \in T_{m_0}M$ (see, e.g., [11, formula (12.4.6)]) and hence $(\mathrm{Hess}\,\mathbf{J})(m_0)(u, u) = 2\mathbf{L}(u)$ for all $u \in T_{m_0}M$. Obviously, if $u \in T_{m_0}F$, both the Hessian and $\mathbf{L}$ vanish. So we need to compute the Hessian on a subspace transversal to $T_{m_0}F$ in order to determine the index of F. Since $T_{m_0}F = T_{m_0}\left(M^{\mathbb{R}/\mathbb{Z}}\right) = (T_{m_0}M)^{\mathbb{R}/\mathbb{Z}}$ is a symplectic vector subspace of $(T_{m_0}M, \omega(m_0))$ (e.g, [16, (2.4.5) and Proposition 4.2.7]), its $\omega(m_0)$-orthogonal complement W is also a symplectic subspace of $(T_{m_0}M, \omega(m_0))$ and we have $T_{m_0}M = (T_{m_0}M)^{\mathbb{R}/\mathbb{Z}} \oplus W$. Thus, we shall compute $(\mathrm{Hess}\,\mathbf{J})(m_0)|_{W\times W}$. The only fixed point of the $\mathbb{R}/\mathbb{Z}$-symplectic representation on W is the origin. We recall the following well-known linear algebra result (see e.g., [6, paragraphs 1 and 2, Section 32, p. 249, 250]).

Lemma 8. *The $2k$-dimensional $\mathbb{R}/\mathbb{Z}$-symplectic representation space W splits as a $\omega(m_0)$-orthogonal sum of irreducible representations: $W = \oplus_{j=1}^k W_j$, where $\dim W_j = 2$.*

For any irreducible symplectic representation of $\mathbb{R}/\mathbb{Z}$ on a two-dimensional symplectic vector space $(U, \mathbf{d}q \wedge \mathbf{d}p)$, the associated momentum map has the expression $U \ni (q, p) \mapsto \frac{a}{2}(q^2 + p^2) \in \mathbb{R}$, where $a \in \mathbb{R}$ is the weight of the representation.

Therefore, for any $w_1 + \cdots + w_k \in \oplus_{j=1}^k W_j$, formula (9) implies that

$$\begin{aligned}(\mathrm{Hess}\,\mathbf{J})(m_0)(w_1 + \cdots + w_k, w_1 + \cdots + w_k) &= \sum_{j=1}^k (\mathrm{Hess}\,\mathbf{J})(m_0)(w_j, w_j) = \sum_{j=1}^k 2\mathbf{L}(w_j, w_j)\\ &= \sum_{j=1}^k a_j\left(q_j^2 + p_j^2\right),\end{aligned} \tag{10}$$

where $a_j \in \mathbb{R}$ are the weights of the irreducible $\mathbb{R}/\mathbb{Z}$-representations and (q_j, p_j) are the symplectic coordinates of $w_j \in W_j$, $j = 1, \ldots, k$. From Lemma 8 and (10) it follows that $\mathbf{J}$ is Morse-Bott and that the index of the connected component $F \subset \mathrm{Crit}\,\mathbf{J}$ equals twice the number of the negative weights $a_j \in \mathbb{R}$. Thus the index of F is even.

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