EULER-MACLAURIN FORMULAS VIA DIFFERENTIAL OPERATORS

YOHANN LE FLOCH ÁLVARO PELAYO

Abstract. Recently there has been a renewed interest in asymptotic Euler-MacLaurin formulas, partly due to applications to spectral theory of differential operators. We obtain such formulas for compactly supported smooth functions f on intervals, polygons, and 3-dimensional polytopes Δ, where the coefficients in the asymptotic expansion are sums of differential operators involving only derivatives of f in directions normal to the faces of Δ. Our formulas apply to wedges of any dimension. This paper builds on, and is motivated by, works of Guillemin, Sternberg, and others, in the past ten years.

1. Introduction

Let Δ be a polytope in $\mathbb{R}^n$. Euler-MacLaurin formulas are expressions which may be used to approximate Riemann sums of smooth functions $f\colon \mathbb{R}^n \to \mathbb{R}$ such as

$$\frac{1}{N^n} \sum_{k \in \mathbb{Z}^n \cap N\Delta} f\left(\frac{k}{N}\right), \tag{1}$$

in terms of integrals involving f. Such formulas may be traced back to the work of L. Euler and C. MacLaurin in the first half of the XIX century. Partly due to its connections to other areas of mathematics, this topic has attracted great interest recently, see for instance Berline-Vergne [BeVe07], Brion-Vergne [BrVe97], Cappell-Shaneson [CS95], Guillemin-Sternberg [GuSt07], Guillemin-Sternberg-Weitsman [GSW], Guillemin-Stroock [GuSt08], Guillemin--Wang [GuWa08], Karshon-Sternberg-Weitsman [KSW05], Shaneson [Sh94], Tate [Ta10], and Vergne [Ve13].

In this paper our goal is to obtain explicit Euler-MacLaurin formulas to approximate (1) for compactly supported functions f on wedges in $\mathbb{R}^n$, intervals, polygons, and polytopes Δ in $\mathbb{R}^3$ (Figure 1), where the coefficients in the asymptotic expansion are sums of differential operators involving only derivatives of f in directions normal to the faces of Δ.

The only reason to restrict ourselves to intervals, polygons and 3-polytopes, is because of technical difficulties due to our method of proof. We plan to extend our techniques to higher dimensions in future works. Our formulas apply to wedges (Figure 2) of any dimension. We will recover several of the formulas which Tate [Ta10] obtained for wedges and polytopes of any dimension; our proof method is different from Tate's in the following sense: in

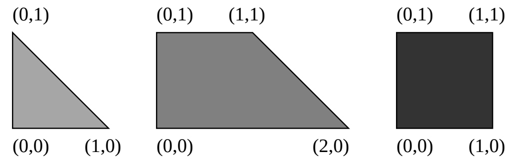

FIGURE 1. Delzant polygons.

[Ta10], Tate begins applying general results of Berline-Vergne [BeVe07] and obtains his formulas through the study of Szasz functions, whereas in our work, we start from a result of Guillemin-Sternberg [GuSt07] and use only elementary freshman calculus to obtain our formulas. Since Guillemin and Sternberg obtained their formulas (also) using only freshman calculus, the proofs (and the general approach) we present in this paper may be considered elementary. In the case of polygons Δ, our approach leads to formulas in which the coefficients in the asymptotic expansion can be explicitly calculated, and we do so in Section 7.

In [ChVN08, Theorem 5.9] Charles and Vũ Ngọc obtained asymptotic expansions for a sum over integral points of a convex polytope, and compute explicitly the first term of this expansion.

Our interest in Euler-MacLaurin formulas originates mainly in the applications they have in spectral theory of differential operators, see for instance the recent articles of Burns-Guillemin-Wang [BuGuWa] and Zelditch [Ze09]. Euler-MacLaurin formulas are also connected to major problems in number theory, see eg. Lagarias [La13].

The structure of the paper is as follows: in Section 2 we will state our main result, and the remaining sections of the paper are devoted to its proof and further refinements.

Victor Guillemin proposed the problem treated in this paper to us as a natural generalization of earlier Euler-Maclaurin results of Guillemin, Sternberg, and others. We would like to enthusiastically thank Professor Guillemin for introducing us to this topic, and for many fruitful discussions which have been important to the paper.

2. Main Result

Let $n \geqslant 1$ and let $\mathbb{Z}^n$ be the integer lattice in $\mathbb{R}^n$. Let $(\mathbb{Z}^n)^*$ and $(\mathbb{R}^n)^*$ be the corresponding dual spaces. Let $\langle \cdot, \cdot \rangle$ denote the pairing of $(\mathbb{R}^n)^*$ with $\mathbb{R}^n$. The subset $W \subset \mathbb{R}^n$ given by the inequalities

$$\langle u_i, x \rangle \leqslant c_i, \quad i \in \{1, \dots, m\} \tag{2}$$

is called an *integer m-wedge* if for every $i \in \{1, \dots, m\}$ the constant c_i is an integer and the vector u_i is a primitive lattice vector in $(\mathbb{R}^n)^*$. Let U be the subspace of $(\mathbb{R}^n)^*$ spanned by the u_i's. We say that W is *regular* if $\{u_1, \dots, u_m\}$ is a lattice basis of the lattice $U \cap (\mathbb{Z}^n)^*$.

When $m = n$, define the diffeomorphism $\varphi \in C^\infty(\mathbb{R}^n, \mathbb{R}^n)$, such that $\varphi(x) = y$ has coordinates

$$y_i := \langle u_i, x\rangle - c_i, \ \ i \in \{1, \dots, n\},$$

in the standard orthonormal basis of $\mathbb{R}^n$. Let $\mathcal{H}_i$ be the facet of W defined by

$$\mathcal{H}_i := \{x \in W \mid \langle u_i, x\rangle = c_i\}\,.$$

For $\alpha = (\alpha_i)_{1\leqslant i\leqslant n} \in \mathbb{N}^n$, set

$$F = \bigcap_{i,\alpha_i>0} \mathcal{H}_i$$

and let $K_\alpha(W)$ be the Jacobian of the diffeomorphism $\varphi_{|F}$, and

$$\int\limits_F^{\star} := K_\alpha(W) \int\limits_F\,.$$

Let $(v_i)_{1\leqslant i\leqslant n}$ be the dual basis of $(u_i)_{1\leqslant i\leqslant n}$. The constant $K_\alpha(W)$ is the volume of the parallelotope formed by the vectors v_i for i such that $\alpha_i = 0$, that is, the primitive outwards vectors defining the face F. We introduce the following notations:

- $\alpha! = \alpha_1! \dots \alpha_n!$;
- $r(\alpha) \in \mathbb{N}^n$ is given by $r(\alpha)_i = 1$ if $\alpha_i > 0$, $r(\alpha)_i = 0$ if $\alpha_i = 0$;
- if $u_1, \dots, u_n \in \mathbb{R}^n$, u_α is the $|\alpha|$-tuple of vectors

$$u_\alpha = (\underbrace{u_1, \dots, u_1}_{\alpha_1 \text{ times}}, \dots, \underbrace{u_n, \dots, u_n}_{\alpha_n \text{ times}});$$

- $\nu(\alpha)$ stands for the number of indices i such that $\alpha_i > 0$.

We define b_n, $n \geqslant 0$ as follows:

$$\begin{aligned} b_0 &= 1; \\ b_1 &= 1/2; \\ b_{2p+1} &= 0 \text{ if } p \geqslant 1; \\ b_{2p} &= (-1)^{p-1} B_p, \end{aligned}$$

with B_p the p-th Bernoulli number. Let

$$C(W, \alpha) := \left(\frac{1}{\alpha!} \prod_{i=1}^{n} b_{\alpha_i}\right) K_\alpha(W).$$

A regular n-wedge is an example of an n-dimensional Delzant polytope. Let $\Delta \subset \mathbb{R}^n$ be an n-dimensional polytope. We say that Δ is *Delzant* if it is a simple and regular polytope[1]. Suppose that Δ has d facets. Then Δ is

[1] Δ is *simple* if there are exactly n edges meeting at each vertex of Δ; it is *regular* if the primitive vectors in the direction of the edges span a basis of $\mathbb{Z}^n$, i.e. for each vertex v of Δ, the edges of Δ which intersect at v lie on rays $v + t\alpha_i$, $0 \leqslant t < \infty$, where α is a lattice basis of $\mathbb{Z}^n$.

defined by d equations: $\langle u_i, x\rangle \leqslant c_i$, where $i \in \{1, \ldots, d\}$. For $q \in [\![1, n]\!]$, we denote by $\mathcal{F}_q$ the set of faces of codimension q of Δ.

The following theorem gives asymptotic Euler-MacLaurin formulas for Riemann sums. It holds in any dimension for wedges, and in dimensions 1, 2, and 3 for polytopes. The case of 4-dimensional polytopes is more complicated to handle with our techniques, and we leave it to future works (see Remark 5.5). The first assertion of the theorem is similar to [Ta10, Proposition 3.1], and the second one is similar to [Ta10, Theorem 5.3].

Theorem 2.1. *Let $f \in \mathrm{C}_0^\infty(\mathbb{R}^n)$. Then the following hold.*

(i) *If W if a regular n-dimensional wedge in $\mathbb{R}^n$,*

$$\frac{1}{N^n} \sum_{k \in \mathbb{Z}^n \cap NW} f\left(\frac{k}{N}\right) \sim \sum_{q \geqslant 0} N^{-q} \sum_{\substack{\alpha \in \mathbb{N}^n \\ |\alpha| = q}} C(W, \alpha) \int_{\bigcap_{\substack{1 \leqslant i \leqslant d \\ \alpha_i > 0}} \mathcal{H}_i} \mathrm{D}^{q - \nu(\alpha)} f \cdot v_{\alpha - r(\alpha)},$$

where the integral is taken over W if the intersection is empty, and the integral over a single point means evaluation at this point. The sign $\sim$ indicates equality modulo $\mathcal{O}(N^{-\infty})$.

(ii) *If $\Delta \subset \mathbb{R}^n$, $n \in \{1, 2, 3\}$, is an n-dimensional Delzant polytope with vertices in $\mathbb{Z}^n$, for every $q \geqslant 1$ and every face $F \in \mathcal{F}_m$ with $m \leqslant q$, there exists a linear differential operator $R_q(F, .)$ of degree $q - m$ depending only on F and involving only derivatives of f in directions normal to F such that*

$$\frac{1}{N^n} \sum_{k \in \mathbb{Z}^n \cap N\Delta} f\left(\frac{k}{N}\right) \sim \int_\Delta f + \sum_{q \geqslant 1} N^{-q} \sum_{\substack{1 \leqslant m \leqslant q \\ F \in \mathcal{F}_m}} \int_F^{\star} R_q(F, f) + \mathcal{O}(N^{-\infty}).$$

The sign $\sim$ indicates equality modulo $\mathcal{O}(N^{-\infty})$.

Theorem 2.1 will follow from combining the upcoming results:

- Proposition 4.1;
- Theorem 5.4;
- Theorem 6.1;

Some of the results of the paper are more general than Theorem 2.1, but we leave them to later sections for simplicity. Our proof of Theorem 2.1 is different from the the proof of Tate's general result in [Ta10, Theorem 5.2], self-contained, and elementary (in the sense that it relies only on freshman calculus). We expect to extend part (ii) of Theorem 2.1 to higher dimensions in future works.

In the case where Δ is a polygon, we give concrete expressions for the coefficients in the formula in Theorem 2.1, see Theorem 7.2.

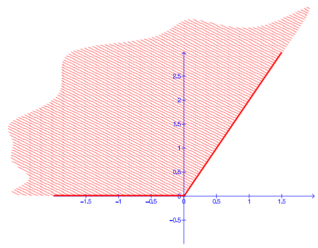

FIGURE 2. A 2-dimensional wedge.

3. Guillemin-Sternberg formulas for regular wedges and Delzant polytopes

3.1. Formula for regular wedges. The following approximation result was recently proven by Guillemin-Sternberg [GuSt07].

Lemma 3.1. *Let W be a regular integer m-wedge defined by (2). Let W_h be the perturbed set defined by $\langle u_i, x\rangle \leqslant c_i + h_i$, $i \in \{1,\ldots,m\}$, where $h = (h_1,\ldots,h_m)$. Then, if $f \in \mathrm{C}_0^\infty(\mathbb{R}^n)$, we have that*

$$\frac{1}{N^n}\sum_{k\in\mathbb{Z}^n\cap NW} f\Big(\frac{k}{N}\Big) \sim \Big(\tau\Big(\frac{1}{N}\frac{\partial}{\partial h}\Big)\int_{W_h} f(x)\,\mathrm{d}x\Big)(h=0),$$

where

$$\tau(s_1,\ldots,s_m) = \tau(s_1)\ldots\tau(s_m)$$

and $\tau(s_i)$ is the Todd function, on the variable s_i, for every $i \in \{1,\ldots,m\}$. The sign $\sim$ indicates equality modulo $\mathcal{O}(N^{-\infty})$.

3.2. Formula for Delzant (i.e. regular and simple) polytopes. Now let $\Delta \subset \mathbb{R}^n$ be an n-dimensional Delzant polytope with vertices in $\mathbb{Z}^n$ and exactly d facets. Then Δ is defined by the inequalities $\langle u^i, x\rangle \leqslant c_i$, $i \in \{1,\ldots,d\}$ where c_i is an integer and $u^i \in (\mathbb{Z}^n)^*$ is a primitive vector perpendicular to the i^{th}-facet of Δ, and pointing outwards from Δ.

Because Δ is simple by assumption, every codimension k face of Δ is defined by a collection of equalities $\langle u^i, x\rangle = c_i$, $i \in F$, where F is a subset of k elements of the set $\{1,\ldots,d\}$. Let W_F denote the k-wedge $\langle u^i, x\rangle \leqslant c_i$, $i \in F$. Because Δ is regular, each k-wedge W_F is regular.

Guillemin and Sternberg have recently shown [GuSt07] the following Euler--MacLaurin formula.

Theorem 3.2. *Let Δ be a Delzant polytope with vertices in $\mathbb{Z}^n$. Let Δ_h be the perturbed polytope defined by the equations $\langle u^i, x\rangle \leqslant c_i + h_i$, $i \in \{1,\ldots,d\}$. Then, if $f \in \mathrm{C}_0^\infty(\mathbb{R}^n)$, we have*

$$\frac{1}{N^n}\sum_{k\in\mathbb{Z}^n\cap N\Delta} f\Big(\frac{k}{N}\Big) \sim \Big(\tau\Big(\frac{1}{N}\frac{\partial}{\partial h}\Big)\int_{\Delta_h} f(x)\,\mathrm{d}x\Big)(h=0),$$

where

$$\tau(s_1,\ldots,s_d) = \tau(s_1)\ldots\tau(s_d)$$

and $\tau(s_i)$ is the Todd function on the variable s_i, for every $i \in \{1, \dots, d\}$.

Remark 3.3. A general asymptotic Euler-MacLaurin formula for Riemann sums over lattice polytopes (simple or not) was given by Tate [Ta10]. As far as we know, Theorem 3.2 does not follow from Tate's formula. Other Euler-MacLaurin formulas have been obtained by Berline-Vergne [BeVe07], Brion--Vergne [BrVe97], Cappell-Shaneson [CS95], Karshon-Sternberg-Weitsman [KSW05], and Zelditch [Ze09] among other authors. ⊘

4. Asymptotic expansion for regular wedges

Let W be a regular integer n-wedge defined by (2). Recall that $(v_i)_{1 \leqslant i \leqslant n}$ is the dual basis of $(u_i)_{1 \leqslant i \leqslant n}$ and $\mathcal{H}_i$ is the facet of W defined by

$$\mathcal{H}_i = \{x \in W \mid \langle u_i, x \rangle = c_i\}.$$

Then v_i generates the edge $\bigcap_{j \neq i} \mathcal{H}_j$. For any integer q, we introduce the operator $T_q(W, .)$ defined by

$$T_q(W, f) = \sum_{\substack{\alpha \in \mathbb{N}^n \\ |\alpha| = q}} \left(\frac{1}{\alpha!} \prod_{i=1}^{n} b_{\alpha_i} \right) K_\alpha(W) \int_{\bigcap_{i, \alpha_i > 0} \mathcal{H}_i} \mathrm{D}^{q - \nu(\alpha)} f \cdot v_{\alpha - r(\alpha)}$$

with the convention that the integral is taken over W if the intersection is empty, and that integrating a function over the vertex means evaluating it at the vertex. $K_\alpha(W)$ is a constant depending only on the face $\bigcap_{i, \alpha_i > 0} \mathcal{H}_i$, of which we gave an interpretation earlier.

Proposition 4.1. *If $f \in \mathrm{C}_0^\infty(\mathbb{R}^n)$, we have that*

$$\frac{1}{N^n} \sum_{k \in \mathbb{Z}^n \cap NW} f\left(\frac{k}{N}\right) \sim \sum_{q \geqslant 0} N^{-q} T_q(f). \tag{3}$$

Proof. The idea of the proof is the following: first, we compute the full asymptotic expansion given by Lemma 3.1 in the case of the standard n-wedge $\{x \in \mathbb{R}^n; x_1 \leqslant 0, \dots, x_n \leqslant 0\}$. Then, we perform a change of variables to deal with the case of a general regular n-wedge.

We start by writing the expansion of

$$\tau(s_1, \dots, s_n) = \tau(s_1) \dots \tau(s_n);$$

we recall that

$$\tau(s) = \frac{s}{1 - \exp(-s)} = \sum_{n=0}^{+\infty} b_n \frac{s^n}{n!}.$$

Write

$$\tau(s_1, \dots, s_n) = \sum_{\alpha \in \mathbb{N}^n} a_\alpha s^\alpha,$$

where $s^\alpha = s_1^{\alpha_1} \dots s_n^{\alpha_n}$. The coefficient S_q of N^{-q} in $\tau\left(\frac{\partial}{\partial h}\right)$ is equal to

$$\sum_{\alpha \in \mathbb{N}^n, |\alpha|=q} a_\alpha \frac{\partial^\alpha}{\partial h^\alpha}$$

with

$$\frac{\partial^\alpha}{\partial h^\alpha} = \frac{\partial^{\alpha_1}}{\partial h_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_n} f}{\partial h_n^{\alpha_n}}.$$

Hence

$$S_q(f) = \sum_{\alpha \in \mathbb{N}^n, |\alpha|=q} \left(\frac{1}{\alpha!} \prod_{i=1}^{n} b_{\alpha_i} \right) \frac{\partial^\alpha}{\partial h^\alpha}.$$

From this result, we deduce a formula for the case of the standard wedge. Remember that

$$\int_{W_h} f(x)\mathrm{d}x = \int_{-\infty}^{h_1} \cdots \int_{-\infty}^{h_n} f(x_1, \dots, x_n)\mathrm{d}x_1 \dots \mathrm{d}x_n;$$

thus

$$\left(\frac{\partial}{\partial h_i} \int_{W_h} f(x)\mathrm{d}x \right)_{|h=0} = \int_{-\infty}^{0} \cdots \int_{-\infty}^{0} f(\widehat{x_i})\widehat{\mathrm{d}x_i} = \int_{\{x_i=0\}} f$$

where $\widehat{x_i} = (x_1, \dots x_{i-1}, 0, x_{i+1}, \dots, x_n)$ and $\widehat{\mathrm{d}x_i} = \mathrm{d}x_1 \dots \mathrm{d}x_{i-1}\mathrm{d}x_{i+1} \dots \mathrm{d}x_n$. From this we obtain a general formula when $\alpha_i \geqslant 1$:

$$\left(\frac{\partial^{\alpha_i}}{\partial h_i^{\alpha_i}} \int_{W_h} f(x)\mathrm{d}x \right)_{|h=0} = \int_{\{x_i=0\}} \frac{\partial^{\alpha_i - 1} f}{\partial x_i^{\alpha_i - 1}}.$$

This finally yields

$$\frac{\partial^\alpha}{\partial h^\alpha} \int_{W_h} f(x)\mathrm{d}x = \int_{\bigcap_{i,\alpha_i>0}\{x_i=0\}} \frac{\partial^{\alpha - r(\alpha)} f}{\partial x^{\alpha - r(\alpha)}}. \tag{4}$$

This gives the desired formula in the case of the standard wedge.

Let us now turn to the general case. Let W be the regular n-wedge defined by (2). Define the diffeomorphism $\varphi : \mathbb{R}^n \to \mathbb{R}^n$, such that $\varphi(x) = y$ has the following coordinates in the standard orthonormal basis of $\mathbb{R}^n$:

$$\forall i \in \{1, \dots, n\} \quad y_i = \langle u_i, x \rangle - c_i.$$

Then $\varphi(W)$ is the standard wedge; moreover, φ is a diffeomorphism from $\bigcap_{i\in I} \mathcal{H}_i$ to $\bigcap_{i\in I}\{x_i = 0\}$ for each subset I of $[\![1, n]\!]$. We have that

$$\sum_{k \in \mathbb{Z}^n \cap NW} f\left(\frac{k}{N}\right) = \sum_{\ell \in \varphi(\mathbb{Z}^n) \cap N\varphi(W)} g\left(\frac{\ell}{N}\right) \overset{c_i \text{ integers}}{=} \sum_{\ell \in \mathbb{Z}^n \cap N\varphi(W)} g\left(\frac{\ell}{N}\right)$$

with $g = f \circ \varphi^{-1}$. But we know from the previous case that

$$\sum_{\ell \in \mathbb{Z}^n \cap N\varphi(W)} g\left(\frac{\ell}{N}\right) \sim \sum_{q \geqslant 0} N^{-q}\, T_q(\varphi(W), g);$$

therefore, it only remains to prove that for every $q \geqslant 0$,

$$T_q(\varphi(W), g) = T_q(W, f).$$

First, we have to express the quantity $\frac{\partial^m g}{\partial y_i^m}$ in terms of f. Since $g = f \circ \varphi^{-1}$, we have

$$\frac{\partial g}{\partial y_i}(y) = \mathrm{D}f(\varphi^{-1}(y)) \cdot \mathrm{D}_{y_i}(\varphi^{-1})(y);$$

but

$$\varphi^{-1}(y) = \sum_{j=1}^{n} (y_j + c_j) v_j,$$

so $\mathrm{D}_{y_i}(\varphi^{-1})(y) = v_i$. It follows that

$$\frac{\partial g}{\partial y_i}(y) = \mathrm{D}f(\varphi^{-1}(y)) \cdot v_i.$$

By induction, we find

$$\frac{\partial^m g}{\partial y_i^m}(y) = \mathrm{D}^m f(\varphi^{-1}(y)) \cdot (v_i, \dots, v_i).$$

Now, we have to understand integrals of the form

$$I = \int\limits_{\bigcap_{i,\alpha_i>0}\{y_i=0\}} \frac{\partial^{\alpha - r(\alpha)} g}{\partial y^{\alpha - r(\alpha)}}.$$

From the previous discussion

$$I = \int\limits_{\varphi\left(\bigcap_{i,\alpha_i>0} \mathcal{H}_i\right)} \mathrm{D}^{|\alpha - r(\alpha)|} f(\varphi^{-1}(y)) \cdot v_{\alpha - r(\alpha)}.$$

Hence

$$I = K_\alpha(W) \int\limits_{\bigcap_{i,\alpha_i>0} \mathcal{H}_i} \mathrm{D}^{q-\nu(\alpha)} f \cdot v_{\alpha - r(\alpha)}$$

where $K_\alpha(W)$ is the Jacobian of the diffeomorphism $\varphi_{|\bigcap_{i,\alpha_i>0} \mathcal{H}_i}$, which was to be proved. □

Another way to write this result is the following: we can eliminate the constants $K_\alpha(W)$ by normalizing the measure on the face $F = \bigcap_{i,\alpha_i>0} \mathcal{H}_i$; we set

$$\int\limits_F^{\star} = K_\alpha(W) \int\limits_F .$$

5. Explicit asymptotic expansion for regular wedges

In order to deduce a formula for polygons and 3-dimensional polytopes, we need to rewrite the asymptotic expansion in Proposition 4.1 in a suitable form. Some of the results of this section apply for general $n \geqslant 1$, and when this is the case we present the most general version of the result. The final results require that we restrict to $n = 1, 2$, or 3.

5.1. **General results.** Recall that for $d \in [\![1, n]\!]$, $\mathcal{F}_d$ denotes the set of faces of codimension d of W. In order to simplify the notations, we set

$$\lambda_\alpha = \frac{1}{\alpha!} \prod_{i=1}^{n} b_{\alpha_i}$$

for $\alpha \in \mathbb{N}^n$. Our objective is to write a formula of the following kind:

$$T_q(W, f) = \sum_{d=0}^{n-1} \sum_{F \in \mathcal{F}_d} S_q(F, f)$$

where $S_q(F, f)$ is a differential operator associated to the face F, with good properties in a sense that we will precise later. Let $F \in \mathcal{F}_d$. There exists a subset $I = \{i_1, \ldots, i_d\}$ of $[\![1, n]\!]$ such that $F = \bigcap_{j \in I} \mathcal{H}_j$; the family $(v_i)_{i \notin I}$ is a basis of the linear subspace spanned by F. A first expression for $S_q(F, f)$ is

$$S_q(F, f) = \sum_{\substack{\alpha \in \mathbb{N}^n, |\alpha| = q \\ \{i,\ \alpha_i > 0\} = I}} \lambda_\alpha \overset{\star}{\int_F} \mathrm{D}^{q-d} f \cdot v_{\alpha - r(\alpha)}.$$

Observe that if $d > q$, then $S_q(F, f) = 0$. If $d = q$, then $S_q(F, f)$ involves the integral

$$\overset{\star}{\int_F} f.$$

When $d < q$, the situation is a little bit more complicated, because we integrate directional derivatives of f involving the vectors $v_i, i \in I$. But we would like to keep only quantities that depend on the face F and nothing else; this is why we decompose the vectors $v_i, i \in I$ as follows:

$$v_i = \sum_{j \notin I} \mu_{ij}^F v_j + \sum_{j \in I} \zeta_{ij}^F n_j \tag{5}$$

where n_j is the outward primitive normal to the facet $\mathcal{H}_j$. Next, we expand the quantity $\mathrm{D}^{q-d} f \cdot v_{\alpha - r(\alpha)}$ as a linear combination of n^{q-d} terms involving the vectors $v_j, j \notin I$ and $n_j, j \in I$.

More precisely, write equation (5) as

$$v_i = \sum_{j=1}^{n} \lambda_{ij}^F w_j$$

where $\lambda_{ij}^F = \mu_{ij}^F$, $w_j = v_j$ if $j \notin I$ and $\lambda_{ij}^F = \zeta_{ij}^F$, $w_j = n_j$ if $j \in I$. Moreover, set $\beta = \alpha - r(\alpha)$ and define integers k_ℓ, $1 \leqslant \ell \leqslant q-d$ as follows: $k_\ell = i_1$ if $1 \leqslant \ell \leqslant \beta_{i_1}$, ..., $k_\ell = i_d$ if $q-d-\beta_{i_d} \leqslant \ell \leqslant q-d$. Then

$$\mathrm{D}^{q-d} f \cdot v_{\alpha - r(\alpha)} = \mathrm{D}^{q-d} f \cdot \left(\sum_{j_1=1}^{n} \lambda_{k_1 j_1}^F w_{j_1}, \ldots, \sum_{j_{q-d}=1}^{n} \lambda_{k_{q-d} j_{q-d}}^F w_{j_{q-d}} \right)$$

which can be written by multilinearity as

$$\mathrm{D}^{q-d} f \cdot v_{\alpha - r(\alpha)} = \sum_{j_1, \ldots, j_{q-d}=1}^{n} \left(\prod_{\ell=1}^{q-d} \lambda_{k_\ell j_\ell}^F \right) \mathrm{D}^{q-d} f \cdot \left(w_{j_1}, \ldots, w_{j_{q-d}} \right).$$

We now want to get rid of the vectors v_j, $j \notin I$ when they appear in the quantity

$$\mathrm{D}^{q-d} f \cdot \left(w_{j_1}, \ldots, w_{j_{q-d}} \right).$$

If $j_1, \ldots, j_{q-d}$ all belong to I, then only the normal vectors n_j appear, and we have nothing to do; this constitutes d^{q-d} favorable cases. In unfavorable cases, we use the following lemma.

Lemma 5.1. *For any function $g \in \mathrm{C}_0^\infty(\mathbb{R}^n)$ and for every $j \notin I$, we have*

$$\int_F^\star \mathrm{D} g \cdot v_j = \int_{F_j}^\star g$$

where F_j is the face of codimension $d+1$ defined by $F_j = F \cap \mathcal{H}_j$.

Proof. Let the elements of $[\![1, n]\!] \setminus I$ be denoted by $x_j, x_{i_{d+2}}, \ldots, x_{i_n}$. One has

$$\begin{aligned}
\int_F^\star \mathrm{D} g \cdot v_j &= \int_{\{x_{i_1} = \cdots = x_{i_d} = 0\}} \mathrm{D} g(\varphi^{-1}(x)) \cdot v_j \; \mathrm{d}x_j \mathrm{d}x_{i_{d+2}} \ldots \mathrm{d}x_{i_n} \\
&= \int_{\{x_{i_1} = \cdots = x_{i_d} = 0\}} \mathrm{D} g(\varphi^{-1}(x)) \cdot \mathrm{D}_{x_j}(\varphi^{-1})(x) \; \mathrm{d}x_j \mathrm{d}x_{i_{d+2}} \ldots \mathrm{d}x_{i_n} \\
&= \int_{\{x_{i_1} = \cdots = x_{i_d} = 0\}} \frac{\partial}{\partial x_j} (g \circ \varphi^{-1})(x) \; \mathrm{d}x_j \mathrm{d}x_{i_{d+2}} \ldots \mathrm{d}x_{i_n} \\
&= \int_{\{x_{i_1} = \cdots = x_{i_d} = x_j = 0\}} (g \circ \varphi^{-1})(x) \; \mathrm{d}x_{i_{d+2}} \ldots \mathrm{d}x_{i_n} = \int_{F_j}^\star g.
\end{aligned}$$

□

5.2. **Results for regular 2 and 3-dimensional wedges.** When $n = 1, 2, 3$, applying Lemma 5.1 to functions of the form

$$g = \mathrm{D}^{q-d-1} f,$$

and repeating this as many times as necessary, we can get rid of all the vectors v_j, $j \notin I$ in the expression of

$$\overset{\star}{\int_F} \mathrm{D}^{q-d} f \cdot v_{\alpha - r(\alpha)},$$

and keep only integrals over faces of codimension greater than d of derivatives of f applied to vectors that are normal to the hyperplanes defining the faces.

Before we state and prove our result, let us introduce a few useful notations. Let $C(W, F)$ be the cone generated by the set

$$\{x - y, y \in W, x \in F\}.$$

If X is a non-empty subset of $\mathbb{R}^n$, let $L(X)$ be the vector subspace generated by the elements of the form $y - x$, $x, y \in X$.

The following two lemmas hold in any dimension (not just 2 and 3).

Lemma 5.2. *Decompose the vectors v_i, $i \in I$, as in equation (5). Choose another set of vectors $(w_i)_{i \in I}$ such that*

- *$\forall i \in I$, w_i belongs to*

$$L\left(\bigcap_{j \in [\![1,n]\!] \setminus \{i\}} \mathcal{H}_j\right) \cap C(W, F),$$

- *the family $((v_j)_{j \notin I}, (w_j)_{j \in I})$ is a primitive lattice basis,*

and write, for $i \in I$,

$$w_i = \sum_{j \notin I} \tilde{\mu}_{ij}^F v_j + \sum_{j \in I} \tilde{\zeta}_{ij}^F n_j.$$

Then for $j \in I$, we have $\tilde{\zeta}_{ij}^F = \zeta_{ij}^F$.

In other words, this means that the scalars ζ_{ij}^F only depend on the face F; when F will be considered as a face of a polytope instead of a wedge, then the contribution coming from each wedge will display the same coefficient.

Proof. By definition of the vectors v_j, n_j, $1 \leqslant j \leqslant n$, one has

$$\forall \ell \in I \qquad \langle v_i, n_\ell \rangle = \sum_{j \notin I} \zeta_{ij}^F \langle n_j, n_\ell \rangle. \tag{6}$$

Hence, the d coefficients ζ_{ij}^F, $j \in I$, are obtained by solving the linear system (6) of d equations. Now, we express the vector w_i in the basis $\mathcal{B} = (v_j)_{1 \leqslant j \leqslant n}$:

$$w_i = \sum_{j \notin I} \alpha_{ij} v_j + \sum_{j \in I} \beta_{ij} v_j.$$

Since the vector w_i belongs to $L(\bigcap_{j\in[\![1,n]\!]\setminus\{i\}} \mathcal{H}_j)$, all the scalar products $\langle w_i, n_j\rangle$, $j \in I \setminus \{i\}$, vanish. This implies that for every $j \in I \setminus \{i\}$, $\beta_{ij} = 0$. Thus, the matrix M of change of basis from $\mathcal{B}$ to

$$\mathcal{B}' = ((v_j)_{j\notin I}, (w_j)_{j\in I})$$

is of the form

$$M = \begin{pmatrix} I_{n-d} & A \\ 0 & B \end{pmatrix}$$

where

$$B = \mathrm{diag}(\beta_{11}, \dots, \beta_{dd}).$$

Since $\mathcal{B}$ and $\mathcal{B}'$ are primitive lattice bases, we have $\det(M) = \pm 1$, and hence for every $i \in I$, $\beta_{ii} = \pm 1$. But v_i and w_i belong to $C(W, F)$, so $\beta_{ii} = 1$. This yields that for $\ell \in I$, we have $\langle w_i, n_\ell\rangle = \langle v_i, n_\ell\rangle$. □

Lemma 5.3. *Decompose the vectors v_i, $i \in I$, as in equation (5), and fix $j \notin I$. Set $J = I \cup \{j\}$, and choose another set of vectors $(w_i)_{i\in J}$ such that*

- *$\forall i \in J$, w_i belongs to*
$$L\left(\bigcap_{k\in[\![1,n]\!]\setminus\{i\}} \mathcal{H}_k\right) \cap C(W, F),$$
- *the family $((v_i)_{i\notin J}, (w_i)_{i\in J})$ is a primitive lattice basis.*

and write, for $i \in I$,

$$w_i = \tilde{\mu}^F_{ij} w_j + \sum_{k\notin J} \tilde{\mu}^F_{ik} v_k + \sum_{k\in I} \tilde{\zeta}^F_{ik} n_k.$$

Then we have $\tilde{\mu}^F_{ij} = \mu^F_{ij}$.

In other words, this means that the scalar μ^F_{ij} only depends on the face $F \cap \mathcal{H}_j$.

Proof. By definition of the vectors v_k, n_k, $1 \leqslant k \leqslant n$, one has

$$\forall k \notin I \qquad \langle v_i, v_k\rangle = \sum_{\ell\notin I} \mu^F_{i\ell} \langle v_\ell, v_k\rangle,$$

which can be written in matrix form

$$A\nu = V \tag{7}$$

where V, ν are the column vectors given by

$$\forall \ell \notin I \qquad V_\ell = \langle v_i, v_\ell\rangle, \quad \nu_\ell = \mu^F_{i\ell}$$

and A is the symmetric matrix whose generic coefficient is

$$A_{k,\ell} = \langle v_k, v_\ell\rangle, \ \ k, \ell \notin I.$$

Similarly, the constants $\tilde{\mu}^F_{i\ell}$ satisfy the system of equations

$$\forall k \notin J \qquad \langle w_i, v_k\rangle = \tilde{\mu}^F_{ij} \langle w_j, v_k\rangle + \sum_{\ell\notin J} \tilde{\mu}^F_{i\ell} \langle v_\ell, v_k\rangle \tag{E_k}$$

and

$$(E_i) \qquad \langle w_i, w_j \rangle = \tilde{\mu}_{ij}^F \|w_j\|^2 + \sum_{\ell \notin J} \tilde{\mu}_{i\ell}^F \langle v_\ell, w_j \rangle.$$

Thanks to the proof of the previous lemma, we know that there exists scalars $\alpha_{k\ell}$, $k \in J$, $\ell \notin J$ such that

$$\forall k \in J \qquad w_k = v_k + \sum_{\ell \notin J} \alpha_{k\ell} v_\ell.$$

Hence we have

$$\langle w_i, w_j \rangle = \langle v_i, v_j \rangle + \sum_{\ell \notin J} \alpha_{j\ell} \langle v_i, v_\ell \rangle + \sum_{k \notin J} \alpha_{ik} \langle v_j, v_k \rangle + \sum_{k,\ell \notin J} \alpha_{ik} \alpha_{j\ell} \langle v_\ell, v_k \rangle$$

as well as

$$\begin{aligned} \forall k \notin J \qquad \langle w_i, v_k \rangle &= \langle v_i, v_k \rangle + \sum_{\ell \notin J} \alpha_{i\ell} \langle v_k, v_\ell \rangle \\ \langle w_j, v_k \rangle &= \langle v_j, v_k \rangle + \sum_{\ell \notin J} \alpha_{j\ell} \langle v_k, v_\ell \rangle \end{aligned}$$

and

$$\|w_j\|^2 = \|v_j\|^2 + 2 \sum_{\ell \notin J} \alpha_{j\ell} + \sum_{k,\ell \notin J} \alpha_{j\ell} \alpha_{jk} \langle v_\ell, v_k \rangle.$$

Using these relations, equations (E_k) become

$$(E'_k) \qquad \langle v_i, v_k \rangle + \sum_{\ell \notin J} \alpha_{i\ell} \langle v_\ell, v_k \rangle = \tilde{\mu}_{ij}^F \langle v_j, v_k \rangle + \sum_{\ell \notin J} \left(\tilde{\mu}_{i\ell}^F + \tilde{\mu}_{ij}^F \alpha_{j\ell} \right) \langle v_\ell, v_k \rangle$$

while equation (E_i) becomes

$$(E'_i) \qquad \begin{aligned} &\langle v_i, v_j \rangle + \sum_{\ell \notin J} \alpha_{j\ell} \langle v_i, v_k \rangle + \sum_{k \notin J} \alpha_{ik} \langle v_j, v_k \rangle + \sum_{k,\ell \notin J} \alpha_{ik} \alpha_{j\ell} \langle v_k, v_\ell \rangle \\ &= \tilde{\mu}_{ij}^F \|v_j\|^2 + 2 \sum_{\ell \notin J} \tilde{\mu}_{ij}^F \alpha_{j\ell} \langle v_j, v_\ell \rangle + \sum_{k,\ell \notin J} \tilde{\mu}_{ij}^F \alpha_{j\ell} \alpha_{jk} \langle v_k, v_\ell \rangle \\ &\qquad + \sum_{\ell \notin J} \tilde{\mu}_{i\ell}^F \langle v_\ell, v_j \rangle + \sum_{k,\ell \notin J} \tilde{\mu}_{i\ell}^F \alpha_{jk} \langle v_k, v_\ell \rangle. \end{aligned}$$

Considering the linear combination (E'_i) - $\sum_{k \notin J} \alpha_{jk}$($E'_k$), we replace equation ($E'_i$) by the new equation (we do no write the details of the computations)

$$\langle v_i, v_j \rangle + \sum_{k \notin J} \alpha_{ik} \langle v_j, v_k \rangle = \tilde{\mu}_{ij}^F \|v_j\|^2 + \sum_{\ell \notin J} \left(\tilde{\mu}_{i\ell}^F + \tilde{\mu}_{ij}^F \alpha_{j\ell} \right) \langle v_j, v_\ell \rangle.$$

Together with equations (E'_k), this means that the coefficients $\tilde{\mu}_{i\ell}^F$ are solutions of the system

$$AU + V = A\tilde{\nu}$$

where A and V are as before, $\tilde{\nu}$ is defined as ν but with the coefficients $\tilde{\mu}_{i\ell}^F$ instead of $\mu_{i\ell}^F$, and U is the column vector whose entries are $U_j = 0$,

$$U_\ell = \alpha_{i\ell} - \tilde{\mu}_{ij}^F \alpha_{j\ell}.$$

Comparing this to (7) yields $\nu = \tilde{\nu} - U$, and in particular $\mu_{ij}^F = \tilde{\mu}_{ij}^F$. □

Theorem 5.4. *Assume that $n \in \{1, 2, 3\}$. For every $q \geqslant 1$ and every face $F \in \mathcal{F}_d$ with $d \leqslant q$, there exists a linear differential operator $R_q(F, .)$ of degree $q - d$ depending only on F (in the sense introduced in the previous lemmas) and involving only derivatives of f in directions normal to the face F such that*

$$T_q(W, f) = \sum_{d=0}^{n-1} \sum_{F \in \mathcal{F}_d} \int_F^{\star} R_q(F, f). \tag{8}$$

Proof. To compute $R_q(F, .)$, we apply the previous technique to faces of codimension smaller than d and gather their contribution as integrals over F. It follows from Lemma 5.2 and Lemma 5.3 that $R_q(F, .)$ depends only on the face F; let us briefly explain how.

If $n = 2$, we have to handle two types of faces: the two edges ($d = 1$) and the vertex ($d = 2$) of the wedge. There is not much to say about the case of the vertex. When we integrate over an edge, and we apply our technique, we will find

- constants μ in front of derivatives of f evaluated at the vertex, and there is nothing to prove;
- constants ζ in front of integrals of derivatives of f on F, and Lemma 5.2 ensures that it only depends on the face F.

If $n = 3$, we have three types of faces, namely planes ($d = 1$), edges ($d = 2$) and the vertex ($d = 3$). The difference with the previous case is that when we consider integrals over a plane, we obtain integrals over edges belonging to this plane, each one displaying a factor μ; Lemma 5.3 ensures that it only depends on the given edge. □

Remark 5.5.

(1) In dimension 4 and higher, the situation is more complicated, and Lemma 5.2 and Lemma 5.3 are not enough to obtain a similar theorem. Indeed, think of the following situation: we take $n = 4$ and want to evaluate

$$I = \int_{\mathcal{H}_1}^{\star} \mathrm{D}^2 f \cdot (v_1, v_1).$$

We start by expanding

$$v_1 = \mu_{12}^{\mathcal{H}_1} v_2 + \mu_{13}^{\mathcal{H}_1} v_3 + \mu_{14}^{\mathcal{H}_1} v_4 + \zeta_{11}^{\mathcal{H}_1} n_1,$$

and we apply Lemma 5.1 to obtain

$$I = \mu_{12}^{\mathcal{H}_1} \int^{\star}_{\mathcal{H}_1 \cap \mathcal{H}_2} \mathrm{D}f \cdot v_1 + \mu_{13}^{\mathcal{H}_1} \int^{\star}_{\mathcal{H}_1 \cap \mathcal{H}_3} \mathrm{D}f \cdot v_1$$
$$+ \mu_{14}^{\mathcal{H}_1} \int^{\star}_{\mathcal{H}_1 \cap \mathcal{H}_4} \mathrm{D}f \cdot v_1 + \zeta_{11}^{\mathcal{H}_1} \int^{\star}_{\mathcal{H}_1} \mathrm{D}^2 f \cdot (v_1, n_1).$$

We have to apply the method one more time for each of these integrals. For instance, we put

$$K = \int^{\star}_{\mathcal{H}_1 \cap \mathcal{H}_2} \mathrm{D}f \cdot v_1$$

and to compute this integral, we write

$$v_1 = \mu_{13}^{\mathcal{H}_1 \cap \mathcal{H}_2} v_3 + \mu_{14}^{\mathcal{H}_1 \cap \mathcal{H}_2} v_4 + \zeta_{11}^{\mathcal{H}_1 \cap \mathcal{H}_2} n_1 + \zeta_{12}^{\mathcal{H}_1 \cap \mathcal{H}_2} n_2$$

which yields, again thanks to Lemma 5.1

$$K = \mu_{13}^{\mathcal{H}_1 \cap \mathcal{H}_2} \int^{\star}_{\mathcal{H}_1 \cap \mathcal{H}_2 \cap \mathcal{H}_3} f + \mu_{14}^{\mathcal{H}_1 \cap \mathcal{H}_2} \int^{\star}_{\mathcal{H}_1 \cap \mathcal{H}_2 \cap \mathcal{H}_4} f +$$
$$\zeta_{11}^{\mathcal{H}_1 \cap \mathcal{H}_2} \int^{\star}_{\mathcal{H}_1 \cap \mathcal{H}_2} \mathrm{D}f \cdot n_1 + \zeta_{12}^{\mathcal{H}_1 \cap \mathcal{H}_2} \int^{\star}_{\mathcal{H}_1 \cap \mathcal{H}_2} \mathrm{D}f \cdot n_2$$

Hence, in the expression of I, we obtain the term

$$\mu_{12}^{\mathcal{H}_1} \mu_{13}^{\mathcal{H}_1 \cap \mathcal{H}_2} \int^{\star}_{\mathcal{H}_1 \cap \mathcal{H}_2 \cap \mathcal{H}_3} f;$$

does the factor only depend on the face $\mathcal{H}_1 \cap \mathcal{H}_2 \cap \mathcal{H}_3$? We think that our previous lemmas are not enough to give an answer to this question.

(2) In principle, one should be able to obtain expressions for the operators $R_q(F, f)$, but this leads to computations involving a very large number of constants, so it is not reasonable to try to write such an expression. However, if we restrict ourselves to dimension 2, we can be fully explicit, as we will see later.

⊘

6. Asymptotic expansion for polygons and 3-dimensional polytopes

As in Section 1, let $\Delta \subset \mathbb{R}^n$ be a Delzant polytope with vertices in $\mathbb{Z}^n$ in dimension $n \in \{1, 2, 3\}$ with equations $\langle u^i, x \rangle \leqslant c_i$, $i \in \{1, \dots, d\}$. Recall that for $m \in [\![1, n]\!]$, $\mathcal{F}_m$ denotes the set of faces of codimension m of Δ.

We introduce as in Theorem 5.4 the operators $R_q(F,.)$ associated to a face F of the polytope (remembering that it only depends on the face as part of the polytope). For any integer q, we define the operator $T_q(\Delta,.)$ by

$$T_q(\Delta, f) = \sum_{m=0}^{n-1} \sum_{F\in\mathcal{F}_m} \int_F^{\star} R_q(F, f). \tag{9}$$

Theorem 6.1. *If $f \in C_0^\infty(\mathbb{R}^n)$, we have that*

$$\frac{1}{N^n} \sum_{k\in\mathbb{Z}^n\cap N\Delta} f\Big(\frac{k}{N}\Big) = \sum_{q\geqslant 0} N^{-q} T_q(\Delta, f).$$

Proof. Notice first that

$$\Delta = \bigcap_{i=1}^{p} W_i$$

where p is the number of vertices of Δ and W_i is the regular wedge which is the intersection of the n facets $\mathcal{H}_j^i$, $1 \leqslant j \leqslant n$, intersecting at the vertex v_i. Cover Δ by open sets Ω_i, $1 \leqslant i \leqslant p$, such that Ω_i contains the vertex v_i and does not intersect any other facet than the $\mathcal{H}_j^i$, $1 \leqslant j \leqslant n$. Choose a partition of unity associated to this open covering and write $f = \sum_{i=1}^p f_i$ where $f_i \in C_0^\infty(\mathbb{R}^n)$ has support included in Ω_i. Then

$$\sum_{k\in\mathbb{Z}^n\cap N\Delta} f\Big(\frac{k}{N}\Big) = \sum_{i=1}^{p} \sum_{k\in\mathbb{Z}^n\cap NW_i} f_i\Big(\frac{k}{N}\Big).$$

Now, from formula (3), we know that for $1 \leqslant i \leqslant d$

$$\sum_{k\in\mathbb{Z}^n\cap NW_i} f_i\Big(\frac{k}{N}\Big) \sim \sum_{\alpha\geqslant 0} N^{-q} T_q(W_i, f_i);$$

hence it is enough to check that for all q

$$\sum_{i=1}^{p} T_q(W_i, f_i) = T_q(\Delta, f).$$

This amounts to show that for each face F of the polytope

$$\sum_{i=1}^{p} R_q(F, f_i) = R_q(F, f).$$

But this is clear because $R_q(F,.)$ is linear and because $\sum_{i=1}^p f_i = f$. □

7. Explicit formula in dimension $n = 2$

We would like to compute explicitly the operators $R_q(\Delta,.)$; unfortunately, as already said, this seems to be quite complicated in all generality. However, we can give nice formulas in dimension 2.

Let Δ be a regular integer polygon defined by (2). In this case, we only have two types of faces: vertices (codimension 2) and edges (codimension

1). Let E (resp. V) be the set of edges (resp. vertices) of Δ. If e belongs to E, let n_e be the associated outward primitive normal vector; if v belongs to V, let $(w_1(v), w_2(v))$ be the integral basis of $\mathbb{Z}$ such that the two edges meeting at v are contained in the half-lines $v + \lambda w_i(v)$, $\lambda \geqslant 0$; we denote by e_i the edge generated by $w_i(v)$. Define the quantities

$$\eta_1(v) = \frac{\langle w_1(v), w_2(v)\rangle}{||w_1(v)||^2}, \qquad \eta_2(v) = \frac{\langle w_1(v), w_2(v)\rangle}{||w_2(v)||^2}.$$

and $\mu(v) = \eta_1(v) + \eta_2(v)$.

Now, let e be an edge, and let $C(\Delta, e)$ be the cone generated by the set $\{x - y, y \in \Delta, x \in e\}$. Given a generator v_1 of $e \cap \mathbb{Z}^2$, there exists a vector $v_2 \in C(\Delta, e) \cap \mathbb{Z}^2$ such that (v_1, v_2) is a primitive lattice basis of $\mathbb{Z}^2$.

Lemma 7.1 (Easy version of Lemma 5.2)**.** *The quantity*

$$\zeta(e) = \frac{\langle v_2, n_e\rangle}{||n_e||^2}$$

does not depend on the choice of $v_2 \in C(\Delta, e) \cap \mathbb{Z}^2$.

Proof. The lemma follows from Lemma 5.2, but its proof is very simple, so we present it next. Choose another vector $w_2 \in C(\Delta, e) \cap \mathbb{Z}^2$ such that (v_1, w_2) is a primitive lattice basis of $\mathbb{Z}^2$. Write $w_2 = \alpha v_1 + \beta v_2$; then, one has $\langle w_2, n_e\rangle = \beta\langle v_2, n_e\rangle$. The matrix of the change of the basis is of the form

$$A = \begin{pmatrix} 1 & \alpha \\ 0 & \beta \end{pmatrix};$$

because its determinant must be ± 1, we have $\beta = \pm 1$. Since both v_2 and w_2 belong to $C(\Delta, e)$, the only possibility is $\beta = 1$. $\square$

Theorem 7.2. *In Theorem 6.1 the operators $T_q(\Delta, .)$ are given by:*

- $T_0(\Delta, f) = \int_\Delta f(x)\mathrm{d}x$;
- $T_1(\Delta, f) = \frac{1}{2}\sum_{e\in E} \int_e^\star f$;
- $T_2(\Delta, f) = \sum_{v\in V} \left(\frac{1}{4} + \frac{\mu(v)}{12}\right) f(v) - \frac{1}{12}\sum_{e\in E} \zeta(e) \int_e^\star \mathrm{D}f \cdot n_e$;
- *if $p > 1$, then*

$$T_{2p}(\Delta, f) = \sum_{e\in E} R_{2p}(e, f) + \sum_{v\in V} R_{2p}(v, f)$$

where

$$R_{2p}(e, f) = (-1)^{p-1}\frac{B_p}{(2p)!}\zeta(e)^{2p-1}\int_e^\star \mathrm{D}^{2p-1} f \cdot (n_e, \dots, n_e)$$

and $R_{2p}(v,f)$ *is equal to*

$$(-1)^{p-2}\sum_{\substack{m+\ell=p\\ m,\ell\geqslant 1}}\frac{B_m B_\ell}{(2m)!(2\ell)!}\mathrm{D}^{2p-2}f(v)\cdot(\underbrace{w_1(v),\dots,w_1(v)}_{2m-1\text{ times}},\underbrace{w_2(v),\dots,w_2(v)}_{2\ell-1\text{ times}})$$

$$+(-1)^{p-2}\frac{B_p}{(2p)!}\eta_1(v)\sum_{k=0}^{2p-2}\zeta(e_1)^k\mathrm{D}^{2p-2}f\cdot(\underbrace{n_{e_1},\dots,n_{e_1}}_{k\text{ times}},\underbrace{w_2(v),\dots,w_2(v)}_{2p-2-k\text{ times}})$$

$$+(-1)^{p-2}\frac{B_p}{(2p)!}\eta_2(v)\sum_{k=0}^{2p-2}\zeta(e_2)^k\mathrm{D}^{2p-2}f\cdot(\underbrace{n_{e_2},\dots,n_{e_2}}_{k\text{ times}},\underbrace{w_1(v),\dots,w_1(v)}_{2p-2-k\text{ times}});$$

- *if* $p>1$, $T_{2p+1}(\Delta,f)$ *is equal to*

$$\frac{(-1)^{p-1}B_p}{2(2p)!}\sum_{v\in V}\left(\mathrm{D}^{2p-1}f(v)\cdot(w_1(v),\dots,w_1(v))+\mathrm{D}^{2p-1}f(v)\cdot(w_2(v),\dots,w_2(v))\right).$$

Remark 7.3. Theorem 7.2 recovers the formula in [Ta10, Corollary 5.4]. To compare the two formulas, one may notice that Tate does not separate the even and odd cases, and that in the odd case nearly every coefficient in Tate's formula vanishes because of the properties of the Bernoulli numbers. ⊘

Proof. We have to compute the operators $R_q(F,.)$ as in Section 5. We start by the case $q=2$. Let v be a vertex and let W be the wedge formed by this vertex and the two incident edges. Define the vectors $w_1(v),w_2(v)$ as before, and let e_1 (resp. e_2) be the edge generated by $w_1(v)$ (resp. $w_2(v)$).

We have

$$T_2(W,f)=\frac{1}{4}f(v)-\frac{1}{12}\left(\int^{\star}_{e_1}\mathrm{D}f\cdot w_2(v)+\int^{\star}_{e_2}\mathrm{D}f\cdot w_1(v)\right)$$

If n_i is the outward primitive vector normal to the edge e_i, we write

$$w_i(v)=\alpha_i w_j(v)+\beta_i n_j$$

where $j=2$ (resp. 1) if $i=1$ (resp. 2). Taking the scalar product with n_j and $w_j(v)$, we find

$$\alpha_i=\frac{\langle w_i(v),w_j(v)\rangle}{||w_j(v)||^2}=\eta_j(v),\qquad \beta_i=\frac{\langle w_i(v),n_j\rangle}{||n_j||^2}=\zeta(e_j).$$

Now, thanks to lemma 5.1, we have (being careful that the vector $w_i(v)$ is the opposite of the vector v_j in this lemma)

$$\int^{\star}_{e_j}\mathrm{D}f\cdot w_i(v)=-\alpha_i f(v)+\zeta(e_j)\int^{\star}_{e_j}\mathrm{D}f\cdot n_j.$$

Adding the contributions from each vertex, we obtain the desired formula.

Now, let $p > 1$; then

$$
\begin{aligned}
T_{2p}(W,f) = (-1)^{p-1}\frac{B_p}{(2p)!}\int_{e_1}^{\star} \mathrm{D}^{2p-1}f\cdot(w_2(v),\dots,w_2(v)) \\
+(-1)^{p-1}\frac{B_p}{(2p)!}\int_{e_2}^{\star} \mathrm{D}^{2p-1}f\cdot(w_1(v),\dots,w_1(v)) \\
+(-1)^{p-2}\sum_{\substack{m+\ell=p,\\ m,\ell\geqslant 1}}\frac{B_m B_\ell}{(2m)!(2\ell)!}\mathrm{D}^{2p-2}f(v)\cdot(\underbrace{w_1(v),\dots,w_1(v)}_{2m-1\text{ times}},\underbrace{w_2(v),\dots,w_2(v)}_{2\ell-1\text{ times}}).
\end{aligned}
$$

We write

$$
\begin{aligned}
\int_{e_j}^{\star} \mathrm{D}^{2p-1}f\cdot(w_i(v),\dots,w_i(v)) = \quad & \eta_j(v)\int_{e_j}^{\star} \mathrm{D}^{2p-1}f\cdot(w_j(v),w_i(v),\dots,w_i(v)) \\
& +\zeta(e_j)\int_{e_j}^{\star} \mathrm{D}^{2p-1}f\cdot(n_j,w_i(v),\dots,w_i(v)).
\end{aligned}
$$

By Lemma 5.1, we have

$$
\int_{e_j}^{\star} \mathrm{D}^{2p-1}f\cdot(w_j(v),w_i(v),\dots,w_i(v)) = -\mathrm{D}^{2p-2}f\cdot(w_i(v),\dots,w_i(v))
$$

and hence we obtain

$$
\begin{aligned}
\int_{e_j}^{\star} \mathrm{D}^{2p-1}f\cdot(w_i(v),\dots,w_i(v)) = -\eta_j(v)\mathrm{D}^{2p-2}f\cdot(w_i(v),\dots,w_i(v)) \\
+\zeta(e_j)\int_{e_j}^{\star} \mathrm{D}^{2p-1}f\cdot(n_j,w_i(v),\dots,w_i(v)).
\end{aligned}
$$

By a straightforward induction, this yields

$$
\begin{aligned}
& \int_{e_j}^{\star} \mathrm{D}^{2p-1}f\cdot(w_i(v),\dots,w_i(v)) \\
= -\eta_j(v)\sum_{k=0}^{2p-2}\zeta(e_j)^k\mathrm{D}^{2p-2}f\cdot(\underbrace{n_j,\dots,n_j}_{k\text{ times}},\underbrace{w_i(v),\dots,w_i(v)}_{2p-2-k\text{ times}}) \\
+\zeta(e_j)^{2p-1}\int_{e_j}^{\star} \mathrm{D}^{2p-1}f\cdot(n_j,\dots,n_j).
\end{aligned}
$$

The case $q = 2p+1$ works in a similar way. □

8. Example

Let us describe an example in the 2-dimensional case.

Let Δ be the triangle with vertices $(0,0)$, $(0,1)$ and $(1,0)$ and let

$$f : \mathbb{R}^2 \to \mathbb{R}$$

be the function defined by

$$f(x_1, x_2) = x_1$$

(multiplied by a cutoff function so that it is compactly supported). Then

$$\mathbb{Z}^2 \cap N\Delta = \{(k_1, k_2) \in \mathbb{Z}^2, \quad 0 \leqslant k_1 \leqslant N, \quad 0 \leqslant k_2 \leqslant N - k_1\}.$$

Therefore, we have

$$\frac{1}{N^2} \sum_{k \in N\Delta \cap \mathbb{Z}^2} f\left(\frac{k}{N}\right) = \frac{1}{N^2} \sum_{k_1=0}^{N} \sum_{k_2=0}^{N-k_1} \frac{k_1}{N} = \frac{1}{N^3} \left((N+1) \sum_{k_1=1}^{N} k_1 - \sum_{k_1=1}^{N} k_1^2. \right).$$

Using standard formulas for sums of integers and squares of integers, one can check that

$$\frac{1}{N^2} \sum_{k \in N\Delta \cap \mathbb{Z}^2} f\left(\frac{k}{N}\right) = \frac{1}{6} + \frac{1}{2N} + \frac{1}{3N^2}.$$

Let us compare this with Theorem 7.2. With the notations of this lemma, we have

$$T_0(\Delta, f) = \int_{\Delta} f = \int_0^1 \left(\int_0^{1-x_1} \mathrm{d}x_2 \right) x_1 \mathrm{d}x_1 = \frac{1}{6}$$

so the zeroth order terms agree. Let us give names to the vertices and edges as follows: we put $v_{13} = (0,0)$, $v_{12} = (0,1)$ and $v_{23} = (1,0)$, and we let e_i denote the edge joining the vertices v_{ij} (or v_{ji}) and v_{ik} (or v_{ki}). Then we have

$$\int_{e_1}^{\star} f = 0, \quad \int_{e_2}^{\star} f = \frac{1}{2}, \quad \int_{e_3}^{\star} f = \frac{1}{2}.$$

and hence

$$T_1(\Delta, f) = 1/2.$$

Furthermore, we have

$$T_2(\Delta, f) = S - T$$

with

$$S = \sum_{v \in V} \left(\frac{1}{4} + \frac{\mu(v)}{12} \right) f(v),$$

and

$$T = \frac{1}{12} \sum_{e \in E} \zeta(e) \int_{e}^{\star} Df \cdot n_e.$$

We have $f(v_{13}) = 0$, $f(v_{12}) = 0$, $f(v_{23}) = 1$. Moreover

$$w_1(v_{23}) = \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \quad w_2(v_{23}) = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

and thus $\mu(v_{23}) = 3/2$. This yields

$$S = 3/8.$$

Now, one can check that

$$\int_{e_1}^{\star} \mathrm{D}f \cdot n_{e_1} = -1, \quad \int_{e_2}^{\star} \mathrm{D}f \cdot n_{e_2} = 1, \quad \int_{e_3}^{\star} \mathrm{D}f \cdot n_{e_3} = 0$$

and $\zeta(e_1) = -1, \quad \zeta(e_2) = -\frac{1}{2}$. We obtain $T = 1/24$ and therefore

$$T_2(\Delta, f) = 1/3.$$

Finally, we have

$$T_q(\Delta, f) = 0, \quad q \geqslant 2$$

because the derivatives of f of order greater than 2 vanish.

9. Final remarks

We believe that one should be able to prove item (ii) in Theorem 2.1 in dimensions $n \geqslant 4$ with the same elementary method we use in this paper (since a similar result was already stated in Tate's article in any dimension). However, in dimensions $n \geqslant 4$ the computations appear to be complicated (but still elementary) and will be the object of future works (see also Remark 5.5).

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Yohann Le Floch
Institut de Recherches Mathématiques de Rennes
Université de Rennes 1
Campus de Beaulieu
F-35042 Rennes cedex, France
E-mail: yohann.lefloch@univ-rennes1.fr

Álvaro Pelayo
Washington University in St Louis
Mathematics Department
One Brookings Drive, Campus Box 1146
St Louis, MO 63130-4899, USA.
E-mail: apelayo@math.wustl.edu