Atiyah’s Connectivity, Morse Theory and Solution Sets

Lecture 4, Miraflores de la Sierra
V School on Geometry, Mechanics and Control

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Outline of Topics

1. Questions
2. Examples
3. Morse Case
4. Vector Case

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I.1. Question

**Solution Set : Toy Example**

Consider $\mathbb{R}^2$ with coordinates $(x, y)$. 

Yes: $\{x = -1\} \cup \{x = 1\} \cup \{y = 0\}$. 

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Solution Set: Toy Example

Consider $\mathbb{R}^2$ with coordinates $(x, y)$. Is the solution set of $(x^2 - 1)y^2 = 0$ connected in $\mathbb{R}^2$?
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1.2. Another Question

Solution Set: Toy Example

Consider $\mathbb{R}^2$ with coordinates $(x, y)$.
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Is the solution set of $\left( x^2 - 1 \right) \left( y^2 + 1 \right) = 0$ connected?
I.2. Another Question

Solution Set : Toy Example

Consider $\mathbb{R}^2$ with coordinates $(x, y)$.

Is the solution set of $(x^2 - 1)(y^2 + 1) = 0$ connected?

No: $\{x = -1\} \cup \{x = 1\}$
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Solution Set : Toy Example

Consider $\mathbb{R}^2$ with coordinates $(x, y)$.

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No: $\{x = -1\} \cup \{x = 1\}$
I.3. Yet Another Question

More Complicated Solution Set

Consider $\mathbb{R}^4$ with coordinates $(x, y, z, t)$. Is the solution set of

$$
\begin{align*}
x^2 + x^2y^2z − xyz^3 − t &= 0 \\
2x + y − t^3 + xyzt + e^t &= 1 \\
x − (2t + 1)(x − z + 1)^3 &= −1
\end{align*}
$$

connected in $\mathbb{R}^4$?
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\end{align*}
\]

connected in $\mathbb{R}^4$? At least one solution $(0, 0, 0, 0)$. 
Let $X \subset \mathbb{R}^m$ defined by $n$ equations. Is $X$ connected?

**Conclusion**

- Easier to check if $n = 2$ from a picture
Let $X \subset \mathbb{R}^m$ defined by $n$ equations. Is $X$ connected?

**Conclusion**

- Easier to check if $n = 2$ from a picture.
- Harder to check if $n = 2$ from equations:

\[
\begin{cases}
    x^2 + e^{x^2y^2} - 1 + \ln(x^2 + 1) &= 1 \\
    x^3 - y^3 &= 3
\end{cases}
\]
I.4. Conclusion

Let $X \subset \mathbb{R}^m$ defined by $n$ equations. Is $X$ connected?

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- If $n \geq 2$ generally extremely difficult to check
Let $X \subset \mathbb{R}^m$ defined by $n$ equations. Is $X$ connected?

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- If $n \geq 2$ generally extremely difficult to check
- So: we need to develop mathematical tools
1.5. Framework: Two Equivalent Questions

**Question A**

Consider functions $f_1, \ldots, f_n : M \subseteq \mathbb{R}^m \to \mathbb{R}$ and $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$, where $M$ is a connected manifold. Is the solution set $\mathcal{S} \subseteq \mathbb{R}^m$ of

\[
\begin{align*}
    f_1(x_1, \ldots, x_m) &= \lambda_1 \\
    \vdots \\
    f_n(x_1, \ldots, x_m) &= \lambda_n
\end{align*}
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a connected subset of $M$?
I.5. Framework: Two Equivalent Questions

Question A
Consider functions $f_1, \ldots, f_n : M \subseteq \mathbb{R}^m \rightarrow \mathbb{R}$ and $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$, where $M$ is a connected manifold. Is the solution set $S \subseteq \mathbb{R}^m$ of

$$\begin{cases} f_1 (x_1, \ldots, x_m) = \lambda_1 \\ \vdots \\ f_n (x_1, \ldots, x_m) = \lambda_n \end{cases}$$

a connected subset of $M$?

Question B (equivalent to A)
Are the fibers of the map $F : M \subseteq \mathbb{R}^m \rightarrow \mathbb{R}^n$

$$F(x_1, \ldots, x_m) := (f_1(x_1, \ldots, x_m), \ldots, f_n(x_1, \ldots, x_m))$$

connected?
1.6. Examples

Scalar-valued function

Are the fibers of \( f : \mathbb{R}^2 \to \mathbb{R} \)

\[
f(x, y) = x^2 + y^2
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Scalar-valued function

- Are the fibers of $f : \mathbb{R}^2 \rightarrow \mathbb{R}$

$$f(x, y) = x^2 + y^2$$

connected?

- Yes, all of them, circles centered at the origin of radius $r \geq 0$
## 1.6. Examples

### Scalar-valued function
- Are the fibers of $f : \mathbb{R}^2 \to \mathbb{R}$
  
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### Vector-valued functions
- Are the fibers of $f : \mathbb{R}^2 \to \mathbb{R}^2$
  
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  connected?
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connected?
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### 1.6. Examples

**Scalar-valued function**
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- Connected?
- **Yes**, all of them, circles centered at the origin of radius \( r \geq 0 \)

**Vector-valued functions**
- Are the fibers of \( f : \mathbb{R}^2 \to \mathbb{R}^2 \)
  \[
  f(x, y) = (x^2, y)
  \]
- Connected?
- **Depends**: \( f^{-1}(c, c') = \{(\pm \sqrt{c}, c')\} \) connected \( \iff \ c = 0 \)
I.7. Connectivity is a very unstable notion: easy observations

**Warnings**

- Small perturbations lead to loss/gain of connectivity

![Diagram of connected sets]

- Intersections of connected sets may be disconnected

![Diagram of disconnected set]

- **So**: Methods to detect connectivity must be **subtle**
II.1. Fibers of Scalar-Valued Function

$\lambda$

$0$

$-1$

$(0, 0, 1)$

$f(x, y, z) = z$

$(0, 0, -1)$

Natural Question: What is the "essential" difference between these examples?

Answer: $f$ having a saddle point!
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Answer: $f$ having a saddle point!
II.2. Fibers of Vector-Valued Function

Let $M = S^2 \times \mathbb{R}^2 \subset \mathbb{R}^5$ with coordinates $(x, y, z, u, v)$. Is the solution set $\mathcal{I}$ of

\[
\begin{align*}
2u^2 + 2v^2 + z &= 1 \\
ux + vy &= 0
\end{align*}
\]

connected?

You're thinking of using "Morse theory" to check this .... And you are right!

Goal: give "method" to answer connectivity questions of this type

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II.2. Fibers of Vector-Valued Function

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I.e. is $F^{-1}(1, 0)$ connected, where

$F: S^2 \times \mathbb{R}^2 \to \mathbb{R}^2$, $F(x, y, z, u, v) := (2u^2 + 2v^2 + z, ux + vy)$?
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And you are right! But Morse Theory for which function?

Goal: give “method” to answer connectivity questions of this type
II.3. Warning

Vector valued case cannot be deduced naively from scalar-valued: Fibers of the map $F : M \subseteq \mathbb{R}^m \rightarrow \mathbb{R}^n$

$$F(x_1, \ldots, x_m) := \left(f_1(x_1, \ldots, x_m), \ldots, f_n(x_1, \ldots, x_m)\right)$$

are intersections of fibers of $f_1, \ldots, f_n$
III.1 Scalar Case — Done with Morse-Bott Theory

Figure: Marston Morse
III.2. Morse-Bott Theory

Morse-Bott Function and Index

\[ f : M \rightarrow \mathbb{R} \] is Morse-Bott if \( \text{Crit}(f) = \bigcup C_i \) and Hessian is transversally non-degenerate on each \( C_i \).

Theorem (Fiber Connectivity)

Suppose \( f : M \rightarrow \mathbb{R} \) is Morse-Bott. Suppose \( M \) compact or \( f \) proper. Index, co-indexes \( \neq 1 \).

Then \( f \) has connected fibers.
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III.3. Idea of Proof of Fiber Connectivity (Compact Case)

Call $\mathcal{C}_k$ set of critical points of index $k$

- **Compactness** $\Rightarrow$ flows converge, stable and unstable manifolds $W^s(\mathcal{C}_k)$, isolated critical values $c_0 < \ldots < c_n$ ...
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- **Index** $\neq 1 \Rightarrow$ codim$(M \setminus W^s(\mathcal{C}_0)) \geq 2 \Rightarrow W^s(\mathcal{C}_0)$ connected $\Rightarrow \mathcal{C}_0 = f^{-1}(c_0)$ connected.
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- Let $c_0 < \lambda < c_1$ and $x_0, x_1 \in f^{-1}(\lambda)$. Push down to $\mathcal{C}_0$ by $\nabla f$, connect in $\mathcal{C}_0$, and get path $\gamma: [0, 1] \to M$. 

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Call $C_k$ set of critical points of index $k$

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- Push $\gamma$ to the inside of $f^{-1}(\lambda)$ by flowing backwards.
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Call \( C_k \) set of critical points of index \( k \)

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- Let \( c_0 < \lambda < c_1 \) and \( x_0, x_1 \in f^{-1}(\lambda) \). Push down to \( C_0 \) by \( \nabla f \), connect in \( C_0 \), and get path \( \gamma: [0, 1] \to M \).

- Push \( \gamma \) to the inside of \( f^{-1}(\lambda) \) by flowing backwards.

- Connectivity of \( f^{-1}(\lambda), \lambda \geq c_1 \), uses similar arguments.
IV.1. Results for Vector-Valued Maps

Figure: Michael Atiyah (London 1929 – )
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Theorem (Atiyah’s Connectivity Theorem, 1982)

Suppose \((M, \omega)\) compact, connected, symplectic, \(m\)-dimensional.
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Suppose \((M, \omega)\) compact, connected, symplectic, \(m\)-dimensional. For \(f_1, \ldots, f_n: M \to \mathbb{R}\), let \(\varphi_i\) flow of \(\mathcal{H}_{f_i}\), where \(\omega(\mathcal{H}_{f_i}, \cdot) = -df_i\).
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Suppose \((M, \omega)\) compact, connected, symplectic, \(m\)-dimensional. For \(f_1, \ldots, f_n: M \to \mathbb{R}\), let \(\varphi_i\) flow of \(\mathcal{H}_{f_i}\), where \(\omega(\mathcal{H}_{f_i}, \cdot) = -df_i\).

Suppose that \(p \mapsto \varphi_1 \circ \ldots \circ \varphi_n(p) \in M\) is a \(\mathbb{T}^n\)-action on \(M\).

Then the fibers of \(F := (f_1, \ldots, f_n): M \to \mathbb{R}^n\) are connected.
IV.2. Sketch of Proof of Atiyah’s Connectivity Theorem

**Statement to prove:** “Fibers of $F$ are connected, for any $m$”
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**Step 1 (T. Frankel, Ann. Math. 1959)**

$f_1$ is a Morse-Bott function with indexes and coindexes $\neq 1$
IV.2. Sketch of Proof of Atiyah’s Connectivity Theorem

Statement to prove: “Fibers of $F$ are connected, for any $m$”

Step 1 (T. Frankel, Ann. Math. 1959)

$f_1$ is a Morse-Bott function with indexes and coindexes $\neq 1$

Step 2 (A mentioned result)

$M_1$ compact and $f_1 : M \to \mathbb{R}$ Morse-Bott $\Rightarrow f_1$ has connected fibers
IV.2. Sketch of Proof of Atiyah’s Connectivity Theorem

Statement to prove: “Fibers of $F$ are connected, for any $m$”

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Step 3
Induction argument, some similarities with Step 1
IV.3. Vector-valued “Integrable Maps"

<table>
<thead>
<tr>
<th>Integrable Systems</th>
</tr>
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<tbody>
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<td>A map ( F = (f_1, \ldots, f_n): (M, \omega) \to \mathbb{R}^n ) is an integrable system if</td>
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(Recall: \( \mathcal{H}_{f_i} \) defined by \( \omega(\mathcal{H}_{f_i}, \cdot) = -df_i \))
IV.3. Vector-valued "Integrable Maps"

Integrable Systems

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- \( \mathcal{H}_{f_1}, \ldots, \mathcal{H}_{f_n} \) are point-wise almost everywhere l.i.
- \( \forall \ i, j, \ f_i \) invariant along flow of \( \mathcal{H}_{f_j} \)

(Recall: \( \mathcal{H}_{f_i} \) defined by \( \omega(\mathcal{H}_{f_i}, \cdot) = -df_i \))

Connected Components of Fibers

- regular fiber
- focus-focus fiber
- transversally elliptic fiber

elliptic point

\( \mathcal{H}_H = 0 \)

\( \mathcal{H}_J = \mathcal{H}_H = 0 \)
IV.4. A Recent Result for "Integrable" Vector Valued Maps

Theorem (P.-Ratiu-V˜u Ngo.c, Pending Revision)

Suppose:

- $M$ compact, connected
- $F = (f_1, f_2): M \rightarrow \mathbb{R}^2$ integrable, non-hyperbolic
- regular bifurcation $\Sigma_{\text{reg}}$ may be deformed smoothly so it does not have horizontal tangencies.

Then $F$ has connected fibers.

Proof "should be" extendable to $\mathbb{R}^n$.
IV.4. A Recent Result for "Integrable" Vector Valued Maps

**Theorem (P.-Ratiu-Vũ Ngọc, Pending Revision)**

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IV.4. A Recent Result for "Integrable" Vector Valued Maps

Theorem (P.-Ratiu-Vũ Ngọc, Pending Revision)

Suppose:

- $M$ compact, connected
- $F = (f_1, f_2): M \to \mathbb{R}^2$ integrable, non-hyperbolic
- regular bifurcation $\Sigma_{\text{reg}}$ may be deformed smoothly so it does not have horizontal tangencies.
IV.4. A Recent Result for "Integrable" Vector Valued Maps

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Proof “should be” extendable to $\mathbb{R}^n$
The fibers of

\[ F: S^2 \times \mathbb{R}^2 \to \mathbb{R}^2, \ F(x,y,z,u,v) := (2u^2 + 2v^2 + z, ux + vy) \]

are, by the previous Theorem, connected.
Applications: Topology

**Connectivity:** probably most basic *topological* question to ask about solution set (after whether \( = \emptyset \)).
### IV.6. Why should we care?

#### Applications: Topology

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#### Applications: Mirror Symmetry, Symplectic Topology

- Integrable system $F: M \to \mathbb{R}^n$ is Lagrangian fibration (LF).
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- Integrable system $F: M \to \mathbb{R}^n$ is Lagrangian fibration (LF).
- LF key in *Mirror Symmetry* and *Symplectic Topology*.
- Fiber connectivity (usually) assumed in theorems about LF.
- We give method to test *whether* a theorem applies to a LF.
IV.7. Idea of Proof of (Preliminary) Theorem

Step 1: Constructing proper Morse-Bott function

Build $\tilde{f}_1 := g(f_1, f_2): M \to \mathbb{R}$ proper, Morse-Bott, index/coindex $\neq 1$

$\Downarrow$

$\tilde{f}_1$ has connected fibers.
IV.7. Idea of Proof of (Preliminary) Theorem

Step 1: Constructing proper Morse-Bott function

Build \( \tilde{f}_1 := g(f_1, f_2) : M \to \mathbb{R} \) proper, Morse-Bott, index/coindex \( \neq 1 \)

\[
\downarrow
\]

\( \tilde{f}_1 \) has connected fibers.

Step 2: from component connectivity to global connectivity

Use fiber connectivity of \( \tilde{f}_1 \) + Integrable systems

\[
\downarrow
\]

fibers of \( F \) are connected
Lecture 4 Summary

- Solution sets $\iff$ fibers of maps
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- Solution sets $\iff$ fibers of maps
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## Lecture 4 Summary

- Solution sets $\iff$ fibers of maps
- Morse-Bott functions, index, proper maps
- Atiyah’s connectivity theorem

### Lecture 4 References

- D. McDuff and D. Salamon: *Introduction to Symplectic Topology* (Chapter 5), Oxford University Press
Summary and References for Lecture 4

Lecture 4 Summary

- Solution sets $\iff$ fibers of maps
- Morse-Bott functions, index, proper maps
- Atiyah’s connectivity theorem
- Connectivity for integrable systems

References:
- J. Milnor: Morse theory
  Princeton University Press 1963
- R. Bott: Morse theory indomitable
  Publ. Math. de L’IHES 68 99-114
- Á. Pelayo, T.S. Ratiu and S. V˜u Ngo.c: Singular Lagrangian fibrations of integrable systems
  Preprint.
- D. McDuff and D. Salamon: Introduction to Symplectic Topology (Chapter 5)
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The end. THANK YOU!