Symplectic actions of 2-tori on 4-manifolds

Alvaro Pelayo

Author address:

University of California–Berkeley, Mathematics Department, 970 Evans Hall # 3840, Berkeley, CA 94720-3840, USA.

E-mail address: `apelayo@math.berkeley.edu`

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Acknowledgements

The author is grateful to Y. Karshon for fruitful discussions about this topic and for her encouragement, as well as for comments on several preliminary versions of this paper, which have enhanced the clarity and accuracy. He is grateful to J.J. Duistermaat for stimulating discussions, specifically on sections 3.3, 3.4 and 4.2, for hospitality on three visits to Utrecht, and for comments on a preliminary version. Moreover, he pointed out a technical omission in a previous version of the proof of Theorem 3.4.3, and which affected the definition of iv) in Definition 7.3.1, and the author is grateful to him for discussions on this matter. He thanks A. Uribe for conversations on symplectic normal forms, and P. Scott for discussions on orbifolds, and helpful feedback and remarks on Chapter 6. He also has benefited from conversations with D. Auroux, D. Burns, P. Deligne, V. Guillemin, A. Hatcher, D. McDuff, M. Pinsonnault, R. Spatzier and E. Zupunski. In particular, E. Zupunski sat through several talks of the author on the paper and offered feedback. The author thanks an anonymous referee for comments which have improved the overall presentation. Additionally, he thanks D. McDuff and P. Deligne for the hospitality during visits to Stony Brook and to IAS in the Winter of 2006, to discuss the content of the article [**12**], which influenced the presentation of some topics in the current article. He thanks Oberlin College for the hospitality during the author's visit (September 2006 – June 2007), while supported by a Rackham Fellowship from the University of Michigan. He also received partial support from an NSF Postdoctoral Fellowship. There are many works related to this paper by a number of authors; the specific study of 4-manifolds with symplectic 2-torus actions was suggested by M. Symington to Y. Karshon, who in turn communicated this question to the author.

Abstract

In this paper we classify symplectic actions of 2-tori on compact connected symplectic 4-manifolds, up to equivariant symplectomorphisms. This extends results of Atiyah, Guillemin-Sternberg, Delzant and Benoist. The classification is in terms of a collection of invariants of the topology of the manifold, of the torus action and of the symplectic form. We construct explicit models of such symplectic manifolds with torus actions, defined in terms of these invariants.

We also classify, up to equivariant symplectomorphisms, symplectic actions of $(2n-2)$-dimensional tori on compact connected $2n$-dimensional symplectic manifolds, when at least one orbit is a $(2n-2)$-dimensional symplectic submanifold. Then we show that a compact connected $2n$-dimensional symplectic manifold (M, σ) equipped with a free symplectic action of a $(2n-2)$-dimensional torus with at least one symplectic orbit is equivariantly diffeomorphic to $M/T \times T$ equipped with the translational action of T. Thus two such symplectic manifolds are equivariantly diffeomorphic if and only if their orbit spaces are surfaces of the same genus.

The paper also contains a description of symplectic actions of a torus T on compact connected symplectic manifolds with at least one $\dim T$-dimensional symplectic orbit, and where the torus is not necessarily $(2n-2)$-dimensional.

CHAPTER 1

Introduction

We extend the theory of Atiyah [**1**], Guillemin [**18**], Guillemin-Sternberg [**19**], Delzant [**10**], and Benoist [**3**] to symplectic actions of tori which are not necessarily Hamiltonian. Although Hamiltonian actions of n-dimensional tori on $2n$-dimensional manifolds are present in many integrable systems in classical mechanics, non-Hamiltonian actions occur also in physics, c.f. Novikov's article [**43**]. Interest on non-Hamiltonian motions may be found in the recent physics literature, for example: Sergi-Ferrario [**51**], Tarasov [**61**] and Tuckerman's articles [**59**], [**60**] and the references therein.

In this paper we give a classification of symplectic actions of 2-tori on compact connected symplectic 4-manifolds in terms of a collection of invariants, some of which are algebraic while others are topological or geometric. A consequence of our classification is that the only compact connected 4-dimensional symplectic manifold equipped with a non-locally-free and non--Hamiltonian effective symplectic action of a 2-torus is, up to equivariant symplectomorphisms, the product $\mathbb{T}^2 \times S^2$, where $\mathbb{T}^2 = (\mathbb{R}/\mathbb{Z})^2$ and the first factor of $\mathbb{T}^2$ acts on the left factor by translations on one component, and the second factor acts on S^2 by rotations about the vertical axis of S^2. The symplectic form is a positive linear combination of the standard translation invariant form on $\mathbb{T}^2$ and the standard rotation invariant form on S^2.

Duistermaat and the author showed in [**12**] that a compact connected symplectic manifold (M, σ) with a symplectic torus action with at least one coisotropic principal orbit is an associated G-bundle $G \times_H M_{\mathrm{h}}$ whose fiber is a symplectic toric manifold M_{h} with T_{h}-action and whose base G/H is a torus bundle over a torus. Here T_{h} is the unique maximal subtorus of T which acts in a Hamiltonian fashion on M, G is a two-step nilpotent Lie group which is an extension of the torus T, and H is a commutative closed Lie subgroup of G which acts on M_{h} via T_{h} and is defined in terms of the holonomy of a certain connection for the principal bundle $M_{\mathrm{reg}} \to M_{\mathrm{reg}}/T$, where M_{reg} is the set of points where the action is free. Precisely, $G = T \times (\mathfrak{l}/\mathfrak{t}_{\mathrm{h}})^*$ where $\mathfrak{l}$ is the kernel of the antisymmetric bilinear form $\sigma^{\mathfrak{t}}$ on $\mathfrak{t}$ which gives the restriction of σ to the orbits, and $\mathfrak{t}_{\mathrm{h}}$ is the Lie algebra of the torus T_{h}. The additive group $(\mathfrak{l}/\mathfrak{t}_{\mathrm{h}})^* \subset \mathfrak{l}^*$, viewed as the set of linear forms on $\mathfrak{l}$ which vanish on $\mathfrak{t}_{\mathrm{h}}$, is the maximal subgroup of $\mathfrak{l}^*$ which acts on the orbit space M/T. The orbit space has a structure of $\mathfrak{l}^*$-parallel space, and as such it is isomorphic to the product of a Delzant polytope and a torus; the torus corresponds to the $(\mathfrak{l}/\mathfrak{t}_{\mathrm{h}})^*$-direction and the Delzant polytope corresponds to a complementary direction $C \simeq \mathfrak{t}_{\mathrm{h}}^*$ in $\mathfrak{l}^*$. The action of $\xi \in (\mathfrak{l}/\mathfrak{t}_{\mathrm{h}})^*$ on the orbit space is defined, using the $\mathfrak{l}^*$-parallel structure, as traveling for time 1 in the direction of ξ.

We then proved that in general, symplectic actions of tori on compact connected symplectic manifolds with at least one coisotropic principal orbit are classified by the antisymmetric bilinear form $\sigma^{\mathfrak{t}}$ on $\mathfrak{t}$, the Hamiltonian torus T_{h}, the momentum polytope associated to T_{h} by the Atiyah-Guillemin-Sternberg theorem, a discrete cocompact subgroup in $(\mathfrak{l}/\mathfrak{t}_{\mathrm{h}})^* \subset \mathfrak{t}^*$ (the period lattice of $(\mathfrak{l}/\mathfrak{t}_{\mathrm{h}})^*$), an antisymmetric bilinear form $c\colon (\mathfrak{l}/\mathfrak{t}_{\mathrm{h}})^* \times (\mathfrak{l}/\mathfrak{t}_{\mathrm{h}})^* \to \mathfrak{l}$ with certain integrality properties (which represents the Chern class of the principal T-bundle $M_{\mathrm{reg}} \to M_{\mathrm{reg}}/T$), and the holonomy invariant of a so called admissible connection for the principal T-bundle $M_{\mathrm{reg}} \to M_{\mathrm{reg}}/T$.

On the other hand suppose that (M, σ) is a compact connected $2n$--dimensional symplectic manifold equipped with an effective action of a $(2n-2)$-dimensional torus T for which at least one T-orbit is a $(2n-2)$-dimensional symplectic submanifold of (M, σ). Then the orbit space M/T is a compact, connected, smooth, orientable orbisurface (2-dimensional orbifold) and the projection mapping $\pi\colon M \to M/T$ is a smooth principal T-orbibundle for which the collection $\{(\mathrm{T}_x(T\cdot x))^{\sigma_x}\}_{x\in M}$ of symplectic orthogonal complements to the tangent spaces to the T-orbits is a flat connection. Let p_0 be any regular point in M/T, $\pi_1^{\mathrm{orb}}(M/T, p_0)$ be the orbifold fundamental group, and $\widetilde{M/T}$ be the orbifold universal cover of M/T. Then the symplectic manifold (M, σ) is T-equivariantly symplectomorphic to the principal T-orbibundle $\widetilde{M/T} \times_{\pi_1^{\mathrm{orb}}(M/T,\, p_0)} T$ with symplectic fibers over the orientable orbisurface M/T, where $\pi_1^{\mathrm{orb}}(M/T, p_0)$ acts on T by means of the monodromy homomorphism of the flat connection, on $\widetilde{M/T}$ by concatenation of paths, and on the product $\widetilde{M/T} \times T$ by the diagonal action. We will describe the symplectic form on this space in Definition 3.3.1. The T-action comes from the T-action on $\widetilde{M/T} \times T$ by translations on the right factor. We also present this construction when T is a torus of any dimension so long as at least one T-orbit is a symplectic submanifold of (M, σ).

Then we will prove that symplectic actions of $(2n-2)$-dimensional tori on compact connected symplectic $2n$-manifolds for which at least one T-orbit is a $(2n-2)$-dimensional symplectic submanifold are classified by a non--degenerate antisymmetric bilinear form $\sigma^{\mathfrak{t}}$ on $\mathfrak{t}$ (the restriction of σ to the orbits), the Fuchsian signature of the orbit space M/T, which is a compact, connected orbisurface (and by this we mean the genus g of the underlying surface and the tuple of orders $\vec{o}$ of the orbifold singularities of M/T), the total symplectic area of M/T, and an element in $T^{2g+n}/\mathcal{G}$ which encodes the holonomy of the aforementioned flat connection for $\pi\colon M \to M/T$, where n is the number of orbifold singular points of M/T, and where $\mathcal{G}$ is the group of matrices

$$\mathcal{G} := \{\begin{pmatrix} A & 0 \\ C & D \end{pmatrix} \in \mathrm{GL}(2g+n,\,\mathbb{Z}) \,|\, A \in \mathrm{Sp}(2g,\,\mathbb{Z}),\ D \in \mathcal{M}\mathrm{S}_n^{\vec{o}}\}.$$

Here $\mathrm{Sp}(2g,\,\mathbb{Z})$ stands for symplectic matrices and $\mathcal{M}\mathrm{S}_n^{\vec{o}}$ for permutation matrices which preserve the tuple $\vec{o}$ of orbifold singularities of M/T.

Moreover we show that if the T-action is free, then M is T-equivariantly diffeomorphic to the product $M/T \times T$, equipped with the action of T by translations on the right factor. Thus two such symplectic manifolds are equivariantly diffeomorphic if and only if their corresponding orbit spaces are surfaces of the same genus.

For dimensional reasons, if the manifold is four-dimensional and the torus is two-dimensional, the antisymmetric bilinear form $\sigma^{\mathfrak{t}}$ can only be trivial or non-degenerate, so either the principal orbits are Lagrangian submanifolds, or they are symplectic submanifolds. Using this fact and the previously mentioned classifications as a stepping stone, we obtain the following classification, a precise and explicit statement of which is Theorem 8.2.1. Let $(M,\,\sigma)$ be a compact connected 4-dimensional symplectic manifold equipped with an effective symplectic action of a 2-torus T. Then one and only one of the following cases occurs:

1) $(M,\,\sigma)$ is a 4-dimensional symplectic-toric manifold, determined by its associated Delzant polygon.
2) $(M,\,\sigma)$ is equivariantly symplectomorphic to a product $\mathbb{T}^2 \times S^2$, where $\mathbb{T}^2 = (\mathbb{R}/\mathbb{Z})^2$ and the first factor of $\mathbb{T}^2$ acts on the left factor by translations on one component, and the second factor acts on S^2 by rotations about the vertical axis of S^2. The symplectic form is a positive linear combination of the standard translation invariant form on $\mathbb{T}^2$ and the standard rotation invariant form on S^2.
3) T acts freely on $(M,\,\sigma)$ with all T-orbits being Lagrangian 2-tori, and $(M,\,\sigma)$ is a principal T-bundle over a 2-torus with Lagrangian fibers. In this case $(M,\,\sigma)$ is classified (as earlier) by a discrete cocompact subgroup P of $\mathfrak{t}^*$, an antisymmetric bilinear mapping $c\colon \mathfrak{t}^* \times \mathfrak{t}^* \to \mathfrak{t}$ which satisfies certain integrality properties, and the so called holonomy invariant of an admissible connection for $M_{\mathrm{reg}} \to M_{\mathrm{reg}}/T$.
4) T acts locally freely on $(M,\,\sigma)$ with all T-orbits being symplectic 2-tori, and $(M,\,\sigma)$ is a principal T-orbibundle over an oriented orbisurface with symplectic fibers. In this case $(M,\,\sigma)$ is classified by an antisymmetric bilinear form $\sigma^{\mathfrak{t}}$ on $\mathfrak{t}$, the Fuchsian signature of M/T, the total symplectic area of M/T, and an element in $T^{2n+g}/\mathcal{G}$, where g is the genus of M/T and n is the number of singular points of M/T.

The paper is organized as follows. In Chapter 2 we describe the structure of the orbit space M/T and study the projection $\pi\colon M \to M/T$, where

(M, σ) has at least one $\dim T$-dimensional symplectic orbit. In Chapter 3 we describe a model of (M, σ) up to T-equivariant symplectomorphisms. In Chapter 4 we describe a model up to T-equivariant diffeomorphisms and provide an alternative model up to T-equivariant symplectomorphisms. In Chapter 5 we classify free symplectic torus actions of a $(2n-2)$-dimensional torus T on $2n$-dimensional symplectic manifolds, when at least one T--orbit is a $(2n-2)$-dimensional symplectic submanifold of (M, σ), up to T-equivariant symplectomorphisms, c.f. Theorem 5.4.1, and also up to T--equivariant diffeomorphisms, c.f. Corollary 5.4.2. In Chapter 6 we study geometric isomorphisms of orbifold homology groups, which is a stepping stone to generalize the classification in Chapter 5 to non-free actions. In Chapter 7 we extend the results in Chapter 5 to non-free actions. We only present those parts of the proofs which are different from the proofs in Chapter 5, and hence we suggest that the paper be read linearly from Chapter 5 to Chapter 7, both included. In turn this approach has the benefit that with virtually no repetition we are able to present the classification in the free case in terms of invariants which are easier to describe than in the general case. In Chapter 8 we provide a classification of symplectic actions of 2-dimensional tori on compact connected 4-dimensional symplectic manifolds. This generalizes the 4-dimensional Delzant's theorem [**10**] to non-Hamiltonian actions. The paper concludes with an appendix in which we briefly present the orbifold theory that we need in the paper.

There is extensive literature on the classification of Hamiltonian, symplectic or smooth torus actions. The papers closest to our paper in spirit are the paper [**10**] by Delzant on the classification of symplectic-toric manifolds (also called Delzant manifolds), and the paper by Duistermaat and the author [**12**] on the classification of symplectic torus actions with coisotropic principal orbits. The following are other contributions related to our work. The paper [**26**] by Karshon on the classification of Hamiltonian circle actions on compact connected 4-dimensional symplectic manifolds. The book of Audin's [**2**] on Hamiltonian torus actions, and Orlik-Raymond's [**44**] and Pao's [**48**] papers, on the classification of actions of 2-dimensional tori on 4-dimensional compact connected smooth manifolds – they do not assume an invariant symplectic structure. Kogan [**31**] studied completely integrable systems with local torus actions. Karshon and Tolman studied centered complexity one Hamiltonian torus actions in arbitrary dimensions in their article [**28**] and Hamiltonian torus actions with 2-dimensional symplectic quotients in [**27**]. McDuff [**40**] and McDuff and Salamon [**41**] studied non-Hamiltonian circle actions, and Ginzburg [**16**] non-Hamiltonian symplectic actions of compact groups under the assumption of a "Lefschetz condition". Symington [**53**] and Leung and Symington [**33**] classified 4-dimensional compact connected symplectic manifolds which are fibered by Lagrangian tori where the fibration may have elliptic or focus-focus singularities.

CHAPTER 2

The orbit space

Unless otherwise stated we assume throughout the chapter that (M, σ) is a compact and connected symplectic manifold and T is a torus which acts effectively on (M, σ) by means of symplectomorphisms. We furthermore assume that at least one T-orbit is a $\dim T$-dimensional symplectic submanifold of (M, σ). We describe the structure of the orbit space M/T, c.f. Definition 2.3.5, and prove that the canonical projection $\pi\colon M \to M/T$ is a principal T-orbibundle endowed with a flat connection, c.f Proposition 2.4.1.

2.1. Symplectic form on the T-orbits

We prove that the symplectic form on every T-orbit of M is given by the same non-degenerate antisymmetric bilinear form.

Let X be an element of the Lie algebra $\mathfrak{t}$ of T, and denote by X_M the smooth vector field on M obtained as the infinitesimal action of X on M. Let ω be a smooth differential form, let L_v denote the Lie derivative with respect to a vector field v, and let $\mathrm{i}_v\,\omega$ denote the usual inner product of ω with v. Since the symplectic form σ is T-invariant, we have that $\mathrm{d}(\mathrm{i}_{X_M}\sigma) = \mathrm{L}_{X_M}\,\sigma = 0$, where the first equality follows by combining $\mathrm{d}\,\sigma = 0$ and the homotopy identity $\mathrm{L}_v = \mathrm{d} \circ \mathrm{i}_v + \mathrm{i}_v \circ \mathrm{d}$. The following result follows from [**12**, Lem. 2.1].

Lemma 2.1.1. *Let (M, σ) be a compact connected symplectic manifold equipped with an effective symplectic action of a torus T for which there is at least one T-orbit which is a $\dim T$-dimensional symplectic submanifold of (M, σ). Then there exists a unique non-degenerate antisymmetric bilinear form $\sigma^{\mathfrak{t}}\colon \mathfrak{t} \times \mathfrak{t} \to \mathbb{R}$ on the Lie algebra $\mathfrak{t}$ of T such that*

$$\sigma_x(X_M(x),\, Y_M(x)) = \sigma^{\mathfrak{t}}(X, Y), \tag{2.1.1}$$

for every X, $Y \in \mathfrak{t}$, and every $x \in M$.

Proof. In [**12**, Lem. 2.1] it was shown that there is a unique antisymmetric bilinear form $\sigma^{\mathfrak{t}}\colon \mathfrak{t} \times \mathfrak{t} \to \mathbb{R}$ on the Lie algebra $\mathfrak{t}$ of T such that expression (2.1.1) holds for every X, $Y \in \mathfrak{t}$, and every $x \in M$. We recall the proof which was given in [**12**]. If u and v are smooth vector fields on M such that $\mathrm{L}_u\,\sigma = 0$ and $\mathrm{L}_v\,\sigma = 0$, then $[u, v]$ is globally a Hamiltonian vector field (associated to $\sigma(u,\, v)$). Indeed, observe that

$$\begin{aligned} \mathrm{i}_{[u,\,v]}\,\sigma &= \mathrm{L}_u(\mathrm{i}_v\,\sigma) - \mathrm{i}_v(\mathrm{L}_u\,\sigma) \\ &= \mathrm{L}_u(\mathrm{i}_v\,\sigma) \\ &= \mathrm{i}_u(\mathrm{d}(\mathrm{i}_v\sigma)) + \mathrm{d}(\mathrm{i}_u(\mathrm{i}_v\,\sigma)) \\ &= \mathrm{d}(\sigma(u,\,v)). \end{aligned} \tag{2.1.2}$$

Applying (2.1.2) to $u = X_M$, $v = Y_M$, where $X,\, Y \in \mathfrak{t}$, we obtain that

$$\mathrm{i}_{[X_M,\,Y_M]}\sigma = \mathrm{d}(\sigma(X_M,\, Y_M)).$$

On the other hand, since T is commutative,

$$[X_M,\, Y_M] = -[X,\, Y]_M = 0.$$

Thus the derivative of the real valued function $x \mapsto \sigma_x(X_M(x),\, Y_M(x))$ identically vanishes on M, which in virtue of the connectedness of M and the fact that σ is a symplectic form, implies expression (2.1.1) for a certain antisymmetric bilinear form $\sigma^{\mathfrak{t}}$ in $\mathfrak{t}$. Since there is a T-orbit of dimension $\dim T$ which is a symplectic submanifold of $(M,\, \sigma)$, the form $\sigma^{\mathfrak{t}}$ must be non-degenerate. □

REMARK 2.1.2. Each tangent space $\mathrm{T}_x(T \cdot x)$ equals the linear span of the vectors $X_M(x)$, $X \in \mathfrak{t}$. The collection of tangent spaces $\mathrm{T}_x(T \cdot x)$ to the T-orbits $T \cdot x$ forms a smooth $\dim T$-dimensional distribution[1], which is integrable, where the integral manifold through x is precisely the T-orbit $T \cdot x$. Since the X_M, $X \in \mathfrak{t}$, are T-invariant vector fields, the distribution $H = \{\mathrm{T}_x(T \cdot x)\}_{x \in M}$ is T-invariant. Each element of H is a symplectic vector space.

2.2. Stabilizer subgroup classification

Recall that if M is an arbitrary smooth manifold equipped with a smooth action of a torus T, for each $x \in M$ we write $T_x := \{t \in T \,|\, t \cdot x = x\}$ for the *stabilizer subgroup of the action of T on M at the point x*. The group T_x is a closed Lie subgroup of T. In this section we study the stabilizer subgroups of the action of T on $(M,\, \sigma)$.

In [**12**, Sec. 2], Duistermaat and the author pointed out that for general symplectic torus actions the stabilizer subgroups of the action need not be connected, which is in contrast with the symplectic actions whose principal orbits are Lagrangian submanifolds, where the stabilizer subgroups are subtori of T; such a fact also may be found as statement (1)(a) in Benoist's article [**3**, Lem. 6.7].

[1]Since T is a commutative group, the Lie brackets of X_M and Y_M are zero for all $X,\, Y \in \mathfrak{t}$, which implies that $\mathcal{D}$ is integrable, which in particular verifies the integrability theorem of Frobenius [**62**, Th. 1.60] for our particular assumptions (although we do *not* need it).

LEMMA 2.2.1. *Let T be a torus. Let (M, σ) be a compact connected symplectic manifold equipped with an effective symplectic action of the torus T, such that at least one T-orbit is a $\dim T$-dimensional symplectic submanifold of (M, σ). Then the stabilizer subgroup of the T-action at every point in M is a finite abelian group.*

PROOF. Let $\mathfrak{t}_x$ denote the Lie algebra of the stabilizer subgroup T_x of the action of T on M at the point x. In the article of Duistermaat and the author [**12**, Lem. 2.2], we observed that for every $x \in M$ there is an inclusion $\mathfrak{t}_x \subset \ker \sigma^{\mathfrak{t}}$, and since by Lemma 2.1.1 $\sigma^{\mathfrak{t}}$ is non-degenerate, its kernel $\ker \sigma^{\mathfrak{t}}$ is trivial, which in turn implies that $\mathfrak{t}_x$ is the trivial vector space, and hence T_x, which is a closed and hence compact subgroup of T, must be a finite group. □

LEMMA 2.2.2. *Let T be a torus. Let (M, σ) be a compact connected symplectic manifold equipped with an effective symplectic action of the torus T such that at least one T-orbit is a $\dim T$-dimensional symplectic submanifold of (M, σ). Then every T-orbit is a $\dim T$-dimensional symplectic submanifold of (M, σ).*

PROOF. Let $x \in M$. Since the torus T is a compact group, the action of T on the smooth manifold M is proper, and the mapping

$$t \mapsto t \cdot x \colon T/T_x \to T \cdot x \tag{2.2.1}$$

is a diffeomorphism, c.f. [**17**, Appendix B] or [**8**, Sec. 23.2], and in particular, the dimension of quotient group T/T_x is equal to the dimension of the T--orbit $T \cdot x$. Since by Lemma 2.2.1 each stabilizer subgroup T_x is finite, the dimension of T/T_x equals $\dim T$, and hence every T-orbit is $\dim T$--dimensional. By Lemma 2.1.1 the symplectic form σ restricted to any T-orbit of the T-action is non-degenerate and hence $T \cdot x$ is a symplectic submanifold of (M, σ). □

COROLLARY 2.2.3. *Let (M, σ) be a compact connected symplectic manifold equipped with an effective symplectic action of a torus T for which at least one, and hence every T-orbit is a $\dim T$-dimensional symplectic submanifold of (M, σ). Then there exists only finitely many different subgroups of T which occur as stabilizer subgroups of the action of T on M, and each of them is a finite group.*

PROOF. By Lemma 2.2.1 every stabilizer subgroup of the action of T on M is a finite group. It follows from the tube theorem of Koszul [**32**], c.f. [**11**, Th. 2.4.1] or [**17**, Th. B24] that in the case of a compact smooth manifold equipped with an effective action of a torus T, there exists only finitely many different subgroups of T which occur as stabilizer subgroups. □

The *principal orbit type of the T-action* is the set of points where the action is free; the *principal T-orbits* are the orbits inside of the principal orbit type.

Corollary 2.2.4. *Let (M, σ) be a compact connected symplectic manifold equipped with an effective symplectic action of a torus T. Then at least one principal T-orbit is a symplectic submanifold of (M, σ) if and only if every principal T-orbit is a symplectic submanifold of (M, σ), if and only if at least one T-orbit is a $\dim T$-dimensional symplectic submanifold of (M, σ), if and only if every T-orbit is a $\dim T$-dimensional symplectic submanifold of (M, σ), if and only if every T-orbit is a symplectic submanifold of (M, σ).*

Proof. The proof follows from Lemma 2.2.2 and the fact that the principal orbit type is always non-empty (open and dense, in fact, see [**11**, Sec. 2.6-2.8]), and hence there exist principal orbits, and these are $\dim T$-dimensional. □

Remark 2.2.5. In [**12**] we used the terminology "coisotropic principal orbits" throughout the paper. In this case there are non-coisotropic orbits of dimension less than $\dim T$, unless the action is free, because the stabilizers are subtori. However, in the case we are treating now, we have seen that if there are symplectic principal orbits, then all orbits are symplectic and of dimension $\dim T$. Keeping this in mind both terminologies are appropriate and make and emphasis on different points.

If M is 4-dimensional and T is 2-dimensional we have the following stronger statement, which follows from the tube theorem, since a finite group acting linearly on a disk must be a cyclic group acting by rotations.

Lemma 2.2.6. *Let T be a 2-torus. Let (M, σ) be a compact connected symplectic 4-manifold equipped with an effective symplectic action of T, such that at least one, and hence every T-orbit is a 2-dimensional symplectic submanifold of (M, σ). Then the stabilizer subgroup of the action of T at every point in M is a cyclic abelian group.*

2.3. Orbifold structure of M/T

We denote the space of all orbits in M of the T-action by M/T, and by $\pi\colon M \to M/T$ the canonical projection. The space M/T, which is called the *orbit space* of the T-action, is provided with the maximal topology for which the canonical projection π is continuous; this topology is Hausdorff. Because M is compact and connected, M/T is compact and connected. For each connected component C of an orbit type $M^H := \{x \in M \mid T_x = H\}$ in M of the subgroup H of T, the action of T on C induces a proper and free action of the torus T/H on C, and $\pi(C)$ has a unique structure of a smooth manifold such that $\pi\colon C \to \pi(C)$ is a principal T/H-bundle. The space M/T is not in general a smooth manifold, c.f. Example 2.3.2. Our next goal is to show that M/T has a natural structure of smooth orbifold. See Section 9.1 for the definition of orbifold that we use.

Example 2.3.1 (Free action). Let (M, σ) be the Cartesian product $(\mathbb{R}/\mathbb{Z})^2 \times S^2$ equipped with the product symplectic form of the standard symplectic (area) form on the torus $(\mathbb{R}/\mathbb{Z})^2$ and the standard area form on the sphere S^2. Let T be the 2-torus $(\mathbb{R}/\mathbb{Z})^2$, and let T act on M by translations on the left factor of the product. Such action of T on M is free, it has symplectic 2-tori as T-orbits, and the orbit space M/T is equal to the 2-sphere S^2.

Probably the simplest example of a 4-dimensional symplectic manifold equipped with a symplectic action of a 2-torus for which the torus orbits are symplectic 2-dimensional tori is the 4-dimensional torus $(\mathbb{R}/\mathbb{Z})^2 \times (\mathbb{R}/\mathbb{Z})^2$ with the standard symplectic form, on which the 2-dimensional torus $(\mathbb{R}/\mathbb{Z})^2$ acts by multiplications on two of the copies of $\mathbb{R}/\mathbb{Z}$ inside of $(\mathbb{R}/\mathbb{Z})^4$. The orbit space is a 2-dimensional torus, so a smooth manifold.

EXAMPLE 2.3.2 (Non-free action). Consider the Cartesian product $S^2 \times (\mathbb{R}/\mathbb{Z})^2$ of the 2-sphere and the 2-torus equipped with the product symplectic form of the standard symplectic (area) form on the torus $(\mathbb{R}/\mathbb{Z})^2$ and the standard area form on the sphere S^2. The 2-torus $(\mathbb{R}/\mathbb{Z})^2$ acts freely by translations on the right factor of the product $S^2 \times (\mathbb{R}/\mathbb{Z})^2$. Consider the action of the finite group $\mathbb{Z}/2\,\mathbb{Z}$ on S^2 which rotates each point horizontally by 180 degrees, and the action of $\mathbb{Z}/2\,\mathbb{Z}$ on the 2-torus $(\mathbb{R}/\mathbb{Z})^2$ given by the antipodal action on the first circle. The diagonal action of $\mathbb{Z}/2\,\mathbb{Z}$ on $S^2 \times (\mathbb{R}/\mathbb{Z})^2$ is free and hence the quotient space $S^2 \times_{\mathbb{Z}/2\,\mathbb{Z}} (\mathbb{R}/\mathbb{Z})^2$ is a smooth manifold. Let (M, σ) be this associated bundle $S^2 \times_{\mathbb{Z}/2\,\mathbb{Z}} (\mathbb{R}/\mathbb{Z})^2$ with the symplectic form and T-actions inherited from the ones given in the product $S^2 \times (\mathbb{R}/\mathbb{Z})^2$, where $T = (\mathbb{R}/\mathbb{Z})^2$. The action of T on M is not free and the T-orbits are symplectic 2-dimensional tori. The orbit space M/T is equal to $S^2/(\mathbb{Z}/2\,\mathbb{Z})$, which is a smooth 2-dimensional orbifold with two singular points of order 2, the South and North poles of S^2.

Let $k := \dim M - \dim T$. By the tube theorem of Koszul, c.f. [**32**], [**11**, Th. 2.4.1], [**17**, Th. B24], for each $x \in M$ there exists a T-invariant open neighborhood U_x of the T-orbit $T \cdot x$ and a T-equivariant diffeomorphism Φ_x from U_x onto the associated bundle $T \times_{T_x} D_x$, where D_x is an open disk centered at the origin in $\mathbb{R}^k = \mathbb{C}^{k/2}$ and T_x acts by linear transformations on D_x. The action of T on $T \times_{T_x} D_x$ is induced by the action of T by translations on the left factor of $T \times D_x$. Because Φ_x is a T-equivariant diffeomorphism, it induces a homeomorphism $\widehat{\Phi}_x$ on the quotient $\widehat{\Phi}_x \colon D_x/T_x \to \pi(U_x)$, and there is a commutative diagram of the form

$$\begin{tikzcd}
T \times D_x \arrow[r, "\pi_x"] & T \times_{T_x} D_x \arrow[r, "\Phi_x"] \arrow[d, "p_x"] & U_x \arrow[d, "\pi|_{U_x}"] \\
D_x \arrow[u, "i_x"] \arrow[r, "\pi'_x"] & D_x/T_x \arrow[r, "\widehat{\Phi}_x"] & \pi(U_x)
\end{tikzcd} , \tag{2.3.1}$$

where π_x, π'_x, p_x are the canonical projection maps, and i_x is the inclusion map. Let

$$\phi_x := \widehat{\Phi}_x \circ \pi'_x. \tag{2.3.2}$$

LEMMA 2.3.3. *Let T be a torus and let Γ, Γ' be finite subgroups of T respectively acting linearly on Γ, Γ'-invariant open subsets D, $D' \subset \mathbb{R}^m$. Let $z \in D$, $z' \in D'$. Let Γ, Γ' act on $T \times D$, $T \times D'$, respectively, by the diagonal action, giving rise to smooth manifolds $T \times_\Gamma D$ and $T \times_{\Gamma'} D'$ equipped with the T-actions induced by the action of T by left translations on $T \times D$ and $T \times D'$. Let $f \colon T \times_\Gamma D \to T \times_{\Gamma'} D'$ be a T-equivariant diffeomorphism such that $f(T \cdot [1, z]_\Gamma) = T \cdot [1, z']_{\Gamma'}$. Then there exist open neighborhoods $U \subset D$ of z, and $U' \subset D'$ of z', and a diffeomorphism $F \colon U \to U'$ which lifts f and such that $F(z) = z'$. The word lift is used in the sense that*

$$\pi_{\Gamma'} \circ i' \circ F = f \circ \pi_\Gamma \circ i,$$

where the maps $i \colon D \to T \times D$, $i' \colon D' \to T \times D'$ are inclusions and the maps $\pi_\Gamma \colon T \times D \to T \times_\Gamma D$, $\pi_{\Gamma'} \colon T \times D' \to T \times_{\Gamma'} D'$ are the canonical projections.

One can obtain Lemma 2.3.3 applying the idea of the proof of [**22**, Lem. 23] by replacing the mapping $f \colon \mathbb{R}^n/\Gamma \to U'/\Gamma'$ therein by the mapping $f \colon T \times_\Gamma D \to T \times_{\Gamma'} D'$.

PROPOSITION 2.3.4. *Let T be a torus. Let (M, σ) be a compact connected symplectic manifold equipped with an effective symplectic action of the torus T for which at least one, and hence every T-orbit, is a $\dim T$-dimensional symplectic submanifold of (M, σ). Then the collection of charts*

$$\widehat{\mathcal{A}} := \{(\pi(U_x),\, D_x,\, \phi_x,\, T_x)\}_{x \in M} \tag{2.3.3}$$

is an orbifold atlas for the orbit space M/T, where for each $x \in M$ the mapping ϕ_x is defined by expression (2.3.2), and the mappings $\widehat{\Phi}_x$, π'_x are defined by diagram (2.3.1).

PROOF. Because $x \in U_x$, we have that $\bigcup_{x \in M} U_x = M$, so the collection $\{\pi(U_x)\}_{x \in M}$ covers M/T. Since U_x is open, $\pi(U_x)$ is open, for each $x \in M$. Let $k := \dim M - \dim T$. By Lemma 2.2.1 the stabilizer group T_x is a finite group of diffeomorphisms. The disks D_x, $x \in M$, given by the tube theorem, are open subsets of $\mathbb{R}^k$, since T_x is a 0-dimensional subgroup of T. Because it is obtained as a composite of continuous maps, ϕ_x in (2.3.2) is continuous and it factors through $\widehat{\Phi}_x \colon D_x/T_x \to \pi(U_x)$, which is the homeomorphism on the bottom part of the right square of diagram (2.3.1).

It is left to show that the mappings ϕ_x, ϕ_y, where x, $y \in M$, are compatible on their overlaps. Indeed, pick $z \in D_x$, $z' \in D_y$ and assume that

$$\phi_x(z) = \phi_y(z'). \tag{2.3.4}$$

Let U_z be an open neighborhood of z such that $\widehat{\Phi}_x(U_z/T_x)$ is contained in the intersection $\pi(U_x) \cap \pi(U_y)$. Let $U_{z'} \subset D_y$ be an open subset of D_y such that $(\widehat{\Phi}_y)^{-1}(\widehat{\Phi}_x(U_z/T_x)) = U_{z'}/T_y$. $U_{z'}$ is an open subset of D_y. Then the composite map

$$\widehat{\Psi}_{xy} := (\widehat{\Phi}_y)^{-1} \circ \widehat{\Phi}_x \colon U_z/T_x \to U_{z'}/T_y \tag{2.3.5}$$

is a homeomorphism which by (2.3.4) satisfies

$$\widehat{\Psi}_{xy}([z]_{T_x}) = [z']_{T_y}. \tag{2.3.6}$$

Since by the tube theorem $\Phi_x \colon T \times_{T_x} D_x \to U_x$ and $\Phi_y \colon T \times_{T_y} D_y \to U_y$ are T-equivariant diffeomorphisms, and by definition $(p_x)^{-1}(U_z/T_x) = T \times_{T_x} U_z$ and $(p_y)^{-1}(U_{z'}/T_y) = T \times_{T_y} U_{z'}$, the composite map

$$\Psi_{xy} := (\Phi_y)^{-1} \circ \Phi_x \colon T \times_{T_x} U_z \to T \times_{T_y} U_{z'} \tag{2.3.7}$$

is a T-equivariant diffeomorphism. By the commutativity of diagram (2.3.1), the map Ψ_{xy} lifts the map $\widehat{\Psi}_{xy}$. Then by (2.3.6), we have that

$$\Psi_{xy}(T \cdot [1,\, z]_{T_x}) = T \cdot [1,\, z']_{T_y}. \tag{2.3.8}$$

Then the map in (2.3.7) is of the form in Lemma 2.3.3 and we can use this lemma to conclude that Ψ_{xy} lifts to a diffeomorphism $\psi_{xy} \colon W_z \to W_{z'}$, where $W_z \subset U_z \subset D_x$ and $W_{z'} \subset U_{z'} \subset D_y$ are open neighborhoods of z, z' respectively, and

$$\psi_{xy}(z) = z'. \tag{2.3.9}$$

Because the map (2.3.7) lifts the map (2.3.5), the diffeomorphism ψ_{xy} lifts the restricted homeomorphism $\widehat{\Psi}_{xy} \colon W_z/T_x \to W_{z'}/T_y$ induced by (2.3.5). Then by (2.3.2) we have that

$$\phi_x \circ \psi_{xy} = \phi_y \tag{2.3.10}$$

on W_z. Expressions (2.3.9) and (2.3.10) precisely describe the compatibility condition of the charts ϕ_x, ϕ_y on their overlaps (see Definition 9.1.1). $\square$

Definition 2.3.5. *Let T be a torus. Let $(M,\, \sigma)$ be a compact connected symplectic manifold equipped with an effective symplectic action of T for which at least one, and hence every T-orbit is a $\dim T$-dimensional symplectic submanifold of $(M,\, \sigma)$. We call $\mathcal{A}$ the class of atlases equivalent to the orbifold atlas $\widehat{\mathcal{A}}$ defined by expression (2.3.3) in Proposition 2.3.4. We denote the orbifold M/T endowed with the class $\mathcal{A}$ by M/T, and the class $\mathcal{A}$ is assumed.*

Remark 2.3.6. Since M is compact and connected, M/T is compact and connected. If $\dim T = 2n-2$, M/T is a compact connected orbisurface. If the T-action is free, then the local groups T_x in Definition 2.3.5 are all trivial, and M/T is a compact connected surface determined up to diffeomorphism by a non-negative integer, its topological genus.

Moreover, since every T-orbit is a $\dim T$-dimensional symplectic submanifold of (M, σ), the action of the torus T on M is a locally free and non-Hamiltonian action. Indeed, by Lemma 2.2.1, the stabilizers are finite, hence discrete groups. On the other hand, if an action is Hamiltonian the T-orbits are isotropic submanifolds, so the T-action cannot be Hamiltonian.

2.4. A flat connection for the projection $M \to M/T$

In this section we prove that the projection $\pi \colon M \to M/T$ onto the orbifold M/T (c.f. Proposition 2.3.4 and Definition 2.3.5) is a smooth principal T-orbibundle (we also use the name "principal T-bundle"). For such orbibundle there are notions of connection and of flat connection, which extend the classical definition for bundles. We show that $\pi \colon M \to M/T$ comes endowed with a flat connection. We refer the reader to Definition 9.1.4 to recall the meaning of these concepts in this setting.

Proposition 2.4.1. *Let $(M,\, \sigma)$ be a compact connected symplectic manifold equipped with an effective symplectic action of a torus T for which at least one, and hence every T-orbit is a $\dim T$-dimensional symplectic submanifold of $(M,\, \sigma)$. Then the collection $\Omega = \{\Omega_x\}_{x \in M}$ of subspaces $\Omega_x \subset \mathrm{T}_x M$, where Ω_x is the σ_x-orthogonal complement to $\mathrm{T}_x(T \cdot x)$ in $\mathrm{T}_x M$, for every $x \in M$, is a smooth distribution on M. The projection mapping $\pi \colon M \to M/T$ is a smooth principal T-orbibundle for which Ω is a T-invariant flat connection.*

Proof. In our particular case, to show that π is smooth principal T--orbibundle amounts to check that for every $z \in M/T$ the following holds: if $\{(\pi(U_x),\, D_x,\, \phi_x,\, T_x)\}_x$ is as in (2.3.3), for each $x \in M$ there exists a T_x-invariant open subset $\widetilde{U}_x$ of D_x and a map $\psi_x \colon T \times \widetilde{U}_x \to Y$ which induces a T-equivariant diffeomorphism between $T \times_{T_x} \widetilde{U}_x$, with the T-action on the left factor, and $p^{-1}(\phi_x(\widetilde{U}_x))$ such that $p \circ \psi_x = \phi_x \circ \pi_2$, where $\pi_2 \colon T \times \widetilde{U}_x \to \widetilde{U}_x$ is the canonical projection. Here T_x acts on $\widetilde{U}_x$ linearly and on $T \times \widetilde{U}_x$ by the diagonal action.

Let, for each $x \in M$, $\widetilde{U}_x = D_x$, and

$$\psi_x(t,\, z) := \Phi_x([t,\, z]_{T_x}).$$

With these choices of $\widetilde{U}_x$, ψ_x, by diagram (2.3.1) and by construction of the orbifold atlas on M/T, c.f. Proposition 2.3.4 and Definition 2.3.5, π satisfies the conditions above and hence it is a smooth principal T-orbibundle.

Because the tangent space to each T-orbit $(\mathrm{T}_x(T \cdot x),\, \sigma|_{\mathrm{T}_x(T\cdot x)})$ is a symplectic vector space, its symplectic orthogonal complement $(\Omega_x,\, \sigma|_{\Omega_x})$ is a symplectic vector space. Here $\sigma|_{\Omega_x}$, $\sigma|_{\mathrm{T}_x(T\cdot x)}$ are the symplectic forms respectively induced by σ on Ω_x, $\mathrm{T}_x(T\cdot x)$. Consider the disk bundle $T \times_{T_x} \Omega_x$ where T_x acts by the induced linearized action on Ω_x, and on $T \times \Omega_x$ by the diagonal action. The translational action of T on the left factor of $T \times \Omega_x$ descends to an action of T on $T \times_{T_x} \Omega_x$. There exists a unique T-invariant symplectic form σ' on $T \times_{T_x} \Omega_x$ such that if $\pi' \colon T \times \Omega_x \to T \times_{T_x} \Omega_x$ is the canonical projection,

$$\pi'^* \sigma' = \sigma|_{\mathrm{T}_x(T\cdot x)} \oplus \sigma|_{\Omega_x}, \tag{2.4.1}$$

where $\sigma|_{\mathrm{T}_x(T\cdot x)} \oplus \sigma|_{\Omega_x}$ denotes the product symplectic form on $T \times \Omega_x$. Then by the symplectic tube theorem of Benoist [**3**, Prop. 1.9], Ortega--Ratiu [**45**], which we use as it was formulated in [**12**, Sec. 11], there exists an open $\mathbb{T}^{1/2(\dim M - \dim T)}$-invariant neighborhood E_x of 0 in Ω_x, an open T-invariant neighborhood V_x of x in M, and a T-equivariant symplectomorphism $\Lambda_x \colon T \times_{T_x} E_x \to V_x$ with $\Lambda_x([1,\, 0]_{T_x}) = x$. By T-equivariance, Λ_x maps the zero section of $T \times_{T_x} E_x$ to the T-orbit $T \cdot x$ through x. It follows from (2.4.1) that the symplectic-orthogonal complement to such section in $T \times_{T_x} E_x$ is precisely the $(\dim M - \dim T)$-dimensional manifold $\pi'(\{1\} \times E_x)$. The composite $\Lambda_x \circ \pi'$ is a local diffeomorphism if E_x is sufficiently small, and hence the image $\Lambda_x(\pi'(\{1\} \times \Omega_x))$ is an integral manifold through x of dimension $\dim M - \dim T$, so $\{\Omega_x\}_{x \in M}$ is a T-invariant integrable distribution. $\square$

Remark 2.4.2. In Proposition 2.3.4 we prove that M/T is a smooth orbifold and nowhere we use that M is symplectic. In other words, we use Koszul's tube theorem instead of its symplectic counterpart due to Benoist and Ortega-Ratiu. This is intentional to emphasize that we do not need M to be symplectic in order to define the orbifold M/T. However, in the proof of Proposition 2.4.1 and later, see for instance the proof of Proposition 3.2.1, we use the charts provided by the symplectic tube theorem, which define an orbifold atlas equivalent to the one defined in Proposition 2.3.4 and hence define the same orbifold structure on M/T, c.f. Definition 2.3.5.

We can formulate Proposition 2.4.1 in the language of foliations as follows.

Corollary 2.4.3. *Let $(M,\, \sigma)$ be a compact connected symplectic manifold equipped with an effective symplectic action of a torus T for which at least one, and hence every T-orbit is a $\dim T$-dimensional symplectic submanifold of $(M,\, \sigma)$. Then the collection of integral manifolds to the symplectic orthogonal complements to the tangent spaces to the T-orbits, c.f. Proposition 2.4.1, is a smooth T-invariant $(\dim M - \dim T)$-dimensional foliation of M.*

2.5. Symplectic tube theorem

For later use, it is convenient that we state the symplectic tube theorem used in the proof of Proposition 2.4.1 in a standard way by replacing Ω_x by $\mathbb{C}^m$ and $\sigma|_{\Omega_x}$ by $\sigma^{\mathbb{C}^m}$, respectively, where $m = 1/2(\dim M - \dim T)$. Indeed, write $\mathrm{i} := \sqrt{-1} \in \mathbb{C}$ and let $\sigma^{\mathbb{C}^m}$ be the symplectic form on $\mathbb{C}^m$

$$\sigma^{\mathbb{C}^m} := \frac{1}{2\,\mathrm{i}} \sum_{j=1}^{m} \mathrm{d}\,\overline{z^j} \wedge \mathrm{d}\, z^j. \tag{2.5.1}$$

Because Ω_x is a symplectic vector space, it has a symplectic basis with $2m$ elements which induces a direct sum decomposition of Ω_x into m mutually $\sigma|_{\Omega_x}$-orthogonal two-dimensional linear subspaces E_j. The stabilizer group T_x acts by means of symplectic linear transformations on the symplectic vector space Ω_x, and the symplectic basis can be chosen so that E_j is T_x-invariant. Averaging any inner product on each E_j over T_x, we obtain a T_x-invariant inner product β_j on E_j, which is unique if we also require that the symplectic inner product of any orthonormal basis with respect to $\sigma|_{\Omega_x}$ is equal to ± 1. This leads to the existence of a unique complex structure on E_j such that, for any unit vector e_j in $(E_j,\, \beta_j)$, we have that $e_j,\, \mathrm{i}\, e_j$ is an orthonormal basis in $(E_j,\, \beta_j)$ and $\sigma|_{\Omega_x}(e_j,\, \mathrm{i}\, e_j) = 1$. This leads to an identification of E_j with $\mathbb{C}$, and hence of Ω_x with $\mathbb{C}^m$, with the symplectic form defined by (2.5.1). The element $c \in \mathbb{T}^m$ acts on $\mathbb{C}^m$ by sending $z \in \mathbb{C}^m$ to the element $c \cdot z$ such that $(c \cdot z)^j = c^j\, z^j$ for every $1 \leq j \leq m$. There is a unique monomorphism of Lie groups $\iota : T_x \to \mathbb{T}^m$ such that $h \in T_x$ acts on $\Omega_x = \mathbb{C}^m$ by sending $z \in \mathbb{C}^m$ to $\iota(h) \cdot z$, hence T_x acts on $T \times \mathbb{C}^m$ by

$$h \star (t,\, z) = (h^{-1} t,\, \iota(h)\, z). \tag{2.5.2}$$

Consider the disk bundle $T \times_{T_x} \mathbb{C}^m$ where T_x acts by (2.5.2). The translational action of T on $T \times \mathbb{C}^m$ descends to an action of T on $T \times_{T_x} \mathbb{C}^m$. By Lemma 2.1.1, the antisymmetric bilinear form $\sigma^{\mathfrak{t}} \colon \mathfrak{t} \times \mathfrak{t} \to \mathbb{R}$ is non--degenerate and hence it determines a unique symplectic form σ^T on T. In view of this, the restricted symplectic form $\sigma|_{\mathrm{T}_x(T \cdot x)} = \sigma^{\mathfrak{t}}$ in the proof of Proposition 2.4.1 does not depend on $x \in M$. The product symplectic form $\sigma^T \oplus \sigma^{\mathbb{C}^m}$ descends to a symplectic form on $T \times_{T_x} E_x$. With this terminology the proof of Proposition 2.4.1 implies the following.

Corollary 2.5.1 (Tube theorem for symplectic orbits). *Let $(M,\, \sigma)$ be a compact connected symplectic manifold equipped with an effective symplectic action of a torus T for which at least one, and hence every T-orbit is a $\dim T$-dimensional symplectic submanifold of $(M,\, \sigma)$. Then there exists an open $\mathbb{T}^m$-invariant neighborhood E_x of the origin in $\mathbb{C}^m$, an open T-invariant neighborhood V_x of x in M, and a T-equivariant symplectomorphism $\Lambda_x \colon T \times_{T_x} E_x \to V_x$ such that $\Lambda_x([1,\, 0]_{T_x}) = x$.*

In Corollary 2.5.1 the product symplectic form is defined pointwise as

$$(\sigma^T \oplus \sigma^{\mathbb{C}^m})_{(t,\, z)}((X,\, u),\, (X',\, u')) = \sigma^{\mathfrak{t}}(X,\, X') + \sigma^{\mathbb{C}^m}(u,\, u').$$

Here we identify each tangent space of the torus T with the Lie algebra $\mathfrak{t}$ of T and each tangent space of a vector space with the vector space itself.

CHAPTER 3

Global model

Unless otherwise stated we assume throughout the chapter that (M, σ) is a compact and connected symplectic manifold and T is a torus which acts effectively on (M, σ) by means of symplectomorphisms. We furthermore assume that at least one T-orbit is a $\dim T$-dimensional symplectic submanifold of (M, σ). We give a model of (M, σ) up to T-equivariant symplectomorphisms.

3.1. Orbifold coverings of M/T

We recall the definition of orbifold covering in Section 9.2.

LEMMA 3.1.1. *Let (M, σ) be a compact connected symplectic manifold equipped with an effective symplectic action of a torus T for which at least one, and hence every T-orbit is a $\dim T$-dimensional symplectic submanifold of M, and let $x \in M$. Let $\pi\colon M \to M/T$ be the canonical projection, and let $\mathcal{I}_x$ be the maximal integral manifold of the distribution Ω of symplectic orthogonal complements to the tangent spaces to the T-orbits which goes through x (c.f. Proposition 2.4.1). Then the inclusion $i_x\colon \mathcal{I}_x \to M$ is an injective immersion between smooth manifolds and the composite $\pi \circ i_x\colon \mathcal{I}_x \to M/T$ is an orbifold covering map.*

PROOF. Since Ω is a smooth distribution, the maximal integral manifold $\mathcal{I}_x$ in injectively immersed in M, c.f. [**62**, Sec. 1]. Let $p = \pi(x) \in M/T$. Then $(\pi \circ i_x)^{-1}(\{p\}) = T \cdot x \cap \mathcal{I}_x$. By Corollary 2.5.1 there is an open $\mathbb{T}^m$-invariant neighborhood E_x of 0 in $\mathbb{C}^m$, an open T-invariant neighborhood V_x of x in M, and a T-equivariant symplectomorphism $\Lambda_x\colon T \times_{T_x} E_x \to V_x$ with $\Lambda_x([1, 0]_{T_x}) = x$. For each $t \in T$, let the mapping $\rho_x(t)\colon E_x \to T \times_{T_x} E_x$ be given by

$$\rho_x(t)(z) = [t, z]_{T_x}. \tag{3.1.1}$$

Since $(\pi \circ i_x)^{-1}(\{p\}) \subset T \cdot x$, there exists a collection $\mathcal{C} := \{t_k\} \subset T$ such that $(\pi \circ i_x)^{-1}(\{p\}) = \{t_k \cdot x \mid t_k \in \mathcal{C}\}$. Since E_x is an open neighborhood of 0 and Λ_x is a T-equivariant symplectomorphism, the image $\Lambda_x(\mathrm{Im}(\rho_x(t_k)))$ is an open neighborhood of $t_k \cdot x$ in $\mathcal{I}_x$, and the image $W_p := \pi(V_x \cap \mathcal{I}_x)$ is an open neighborhood of $p \in M/T$. Let $V(t_k \cdot x) := \Lambda_x(\mathrm{Im}(\rho_x(t_k)))$. For each $t_k \in \mathcal{C}$, the set $V(t_k \cdot x)$ is a connected subset of $\mathcal{I}_x$, since Λ_x and ρ_x are continuous. Each connected component K of $V_x \cap \mathcal{I}_x$ is of the form $K = V(t_k \cdot x)$ for a unique $t_k \in \mathcal{C}$. The definitions of the maps in (2.3.1) imply that $\pi(V(t_k{\cdot}x)) = W_p$ for all $t_k \in \mathcal{C}$. By the commutativity of diagram (2.3.1), the composite mapping

$$\Lambda_x \circ \rho_x(t_k)\colon E_x \to V(t_k \cdot x) \subset \mathcal{I}_x \tag{3.1.2}$$

is a homeomorphism and hence a chart for $\mathcal{I}_x$ around $t_k \cdot x$, for each $t_k \in \mathcal{C}$. Since for each $t_k \in \mathcal{C}$ we have that $\pi(V(t_k \cdot x)) = W_p$, the mapping

$$\psi_x := (\pi \circ i_x) \circ (\Lambda_x \circ \rho_x(1))\colon E_x \to W_p$$

is surjective. Moreover, ψ_x is smooth and factors through the homeomorphism $\widehat{\Lambda}_x\colon E_x/T_x \to W_p$, and hence it is an orbifold chart for M/T around p. The T-equivariance of Λ_x and (3.1.1) imply that $(\pi \circ i_x) \circ (\Lambda_x \circ \rho_x(t_k)) = \psi_x$ for all $t_k \in \mathcal{C}$. By taking the neighborhood W_p around p, the map $\pi \circ i_x$ is an orbifold covering as in Definition 9.2.1, where therein we take $U := W_p$, and the charts in (3.1.2) as charts for $\mathcal{I}_x$. □

We will see in Section 3.2 that the map $\pi \circ i_x$ in Lemma 3.1.1 respects the symplectic forms, where the form on $\mathcal{I}_x$ is the restriction of σ, and the form on M/T is the natural symplectic form that we define therein.

Remark 3.1.2. Let M be an arbitrary smooth manifold. It is a basic fact of foliation theory [**62**, Sec. 1] that in general, the integral manifolds of a smooth distribution D on M are injectively immersed manifolds in M, but they are not necessarily embedded or compact. For example the one-parameter subgroup of tori $\{(t,\, \lambda t) + \mathbb{Z}^2 \in \mathbb{R}^2/\mathbb{Z}^2 \,|\, t \in \mathbb{R}\}$, in which the constant λ is an irrational real number. This is the maximal integral manifold through $(0,\, 0)$ of the distribution which is spanned by the constant vector field $(1,\, \lambda)$, and it is non-compact.

It is an exercise to verify that if $\mathcal{J}_x$ is a maximal integral manifold of a smooth distribution D which passes through x, then $\mathcal{J}_x$ must contain every end point of a smooth curve γ which starts at x and satisfies the condition that for each t its velocity vector $\mathrm{d}\,\gamma(t)/\,\mathrm{d}\,t$ belongs to $D_{\gamma(t)}$, and conversely each such end point is contained in an integral manifold through x. Therefore $\mathcal{J}_x$ is the set of all such endpoints, which is the unique maximal integral manifold through x. It remains to show that this set $\mathcal{J}_x$ is an injectively immersed manifold in M c.f. [**62**, Sec. 1]. In Remark 3.4.5 we give a self-contained proof of this fact for the particular case of our orbibundle $\pi\colon M \to M/T$ in Proposition 2.4.1.

3.2. Symplectic structure on M/T

We prove that M/T comes endowed with a symplectic structure. In the appendix Section 9.3 we recall how to define symplectic structures on orbifolds.

Lemma 3.2.1. *Let $(M,\, \sigma)$ be a compact connected symplectic manifold equipped with an effective (resp. free) symplectic action of a torus T for which at least one, and hence every T-orbit is a symplectic* $\dim T$*-dimensional submanifold of $(M,\, \sigma)$. Then there exists a unique 2-form ν on the orbit* *space M/T such that $\pi^*\nu|_{\Omega_x} = \sigma|_{\Omega_x}$ for every $x \in M$, where $\{\Omega_x\}_{x\in M}$ is the distribution on M of symplectic orthogonal complements to the tangent spaces to the T-orbits, and $\pi\colon M \to M/T$ is the projection map, c.f. Proposition 2.4.1. Moreover, the form ν is symplectic, and so the pair $(M/T,\, \nu)$ is a compact, connected symplectic orbifold (resp. manifold).*

PROOF. Let $m = 1/2(\dim M - \dim T)$. By Corollary 2.5.1, for each $x \in M$ there exists a T-invariant open neighborhood V_x of the T-orbit $T \cdot x$, a $\mathbb{T}^m$-invariant neighborhood E_x of 0 in $\mathbb{C}^m$, and a T-equivariant symplectomorphism Λ_x from $T \times_{T_x} E_x$ with the symplectic form $\widetilde{\sigma^T \oplus \sigma^{\mathbb{C}^m}}$ onto V_x. As in Proposition 2.3.4, the collection of charts $\{(\pi(V_x),\, E_x,\, \phi'_x,\, T_x)\}_{x\in M}$ given by analogy with expression (2.3.3), and where ϕ'_x is defined by analogy with expression (2.3.2), is an orbifold atlas for the orbit space M/T which is equivalent to the one given in Proposition 2.3.4 and hence defines the orbifold structure of M/T given in Definition 2.3.5 (in the general definition of smooth orbifold, c.f. Definition 9.1.1, $\widetilde{U}_i = E_x$, $U_i = \pi'(V_x)$). Because by Remark 9.3.1 it suffices to define a smooth differential form on the charts of any atlas of our choice, the collection $\{\nu_x\}_{x\in M}$ given by $\nu_x := \sigma^{\mathbb{C}^m}$ defines a unique smooth differential 2-form on the orbit space M/T. Because $\sigma^{\mathbb{C}^m}$ is moreover symplectic, each ν_x is a symplectic form, and hence so is ν on M/T. □

The following is an easy consequence of Lemma 3.2.1.

LEMMA 3.2.2. *Let $(M,\, \sigma)$, $(M',\, \sigma')$ be compact connected symplectic manifolds equipped with an effective (resp. free) symplectic action of a torus T for which at least one, and hence every T-orbit is a $\dim T$-dimensional symplectic submanifold of $(M,\, \sigma)$ and $(M',\, \sigma')$, respectively. Suppose additionally that $(M,\, \sigma)$ is T-equivariantly symplectomorphic to $(M',\, \sigma')$. Then the symplectic orbit spaces $(M/T,\, \nu)$ and $(M'/T,\, \nu')$ are symplectomorphic.*

REMARK 3.2.3. Recall the orbifold Moser's theorem, as written by McCarthy and Wolfson [**39**, Th. 3.3]: *Let X be a compact orbifold. Suppose that $\{\rho_t\}$, $0 \le t \le 1$, is a family of orbifold symplectic forms on X such that $[\rho_t] \in \mathrm{H}^2(X,\, \mathbb{R})$ is independent of t. Then there is a family of orbifold diffeomorphisms $g_t\colon X \to X$, $0 \le t \le 1$, such that $g_t^*(\rho_t) = \rho_0$.* As McCarthy and Wolfson point out, the proof is the same as for the classical result since Hodge theory holds.

Under the assumptions of Lemma 3.2.2, if $\dim T = \dim M - 2$, then the total symplectic area of the symplectic orbit space $(M/T,\, \nu)$ equals the total symplectic area of $(M'/T,\, \nu')$. By the orbifold Moser's theorem, if $(M,\, \sigma)$, $(M',\, \sigma')$ are any symplectic manifolds equipped with an action of a torus T of dimension $\dim T = \dim M - 2$, for which at least one and hence every T-orbit is a $\dim T$-dimensional symplectic submanifold of $(M,\, \sigma)$, and such that the symplectic area of the orbit space $(M/T,\, \nu)$ equals the symplectic area of the orbit space $(M'/T,\, \nu')$, and the Fuchsian signature of the orbisurfaces M/T and M'/T are equal, then $(M/T,\, \nu)$ is T-equivariantly (orbifold) symplectomorphic to $(M'/T, \nu')$; we emphasize that this is the case because the orbit spaces M/T and M'/T are 2-dimensional. This result is used in the proof of Theorem 5.4.1 and Theorem 7.4.1.

3.3. Model of (M, σ): definition

In the following, we define a T-equivariant symplectic model for (M, σ): for each regular point $p_0 \in M/T$ we construct a smooth manifold $M_{\mathrm{model},\, p_0}$, a T-invariant symplectic form σ_{model} on $M_{\mathrm{model},\, p_0}$, and an effective symplectic action of the torus T on $M_{\mathrm{model},\, p_0}$. The ingredients for the construction of such model are Proposition 2.4.1 and Lemma 3.2.1. See sections 9.2, 9.3 to recall the terminology on orbifolds we use below.

Definition 3.3.1. *Let (M, σ) be a compact connected symplectic manifold equipped with an effective symplectic action of a torus T for which at least one, and hence every T-orbit is a $\dim T$-dimensional symplectic submanifold of (M, σ). We define the space that we call the* T-equivariant symplectic model $(M_{\mathrm{model},\, p_0}, \sigma_{\mathrm{model}})$ of (M, σ) based at a regular point $p_0 \in M/T$ *as follows.*

i) *The space $M_{\mathrm{model},\, p_0}$ is the associated bundle*

$$M_{\mathrm{model},\, p_0} := \widetilde{M/T} \times_{\pi_1^{\mathrm{orb}}(M/T,\, p_0)} T,$$

where the space $\widetilde{M/T}$ denotes the orbifold universal cover of the orbifold M/T based at a regular point $p_0 \in M/T$, and the orbifold fundamental group $\pi_1^{\mathrm{orb}}(M/T, p_0)$ acts on the Cartesian product $\widetilde{M/T} \times T$ by the diagonal action $x\,(y, t) = (x \star y^{-1}, \mu(x) \cdot t)$, where $\star\colon \pi_1^{\mathrm{orb}}(M/T, p_0) \times \widetilde{M/T} \to \widetilde{M/T}$ denotes the natural action of $\pi_1^{\mathrm{orb}}(M/T, p_0)$ on $\widetilde{M/T}$, and $\mu\colon \pi_1^{\mathrm{orb}}(M/T, p_0) \to T$ denotes the monodromy homomorphism of the flat connection $\Omega := \{\Omega_x\}_{x \in M}$ given by the symplectic orthogonal complements to the tangent spaces to the T-orbits (c.f. Proposition 2.4.1).

ii) *The symplectic form σ_{model} is induced on the quotient by the product symplectic form on the Cartesian product $\widetilde{M/T} \times T$. The symplectic form on $\widetilde{M/T}$ is defined as the pullback by the orbifold universal covering map $\widetilde{M/T} \to M/T$ of the unique 2-form ν on M/T such that $\pi^*\nu|_{\Omega_x} = \sigma|_{\Omega_x}$ for every $x \in M$ (c.f. Lemma 3.2.1). The symplectic form on the torus T is the unique T-invariant symplectic form determined by the non-degenerate antisymmetric bilinear form $\sigma^{\mathfrak{t}}$ such that $\sigma_x(X_M(x), Y_M(x)) = \sigma^{\mathfrak{t}}(X, Y)$, for every $X, Y \in \mathfrak{t}$, and every $x \in M$ (c.f. Lemma 2.1.1). See Remark 3.3.2.*

iii) *The action of T on the space $M_{\mathrm{model},\, p_0}$ is the action of T by translations which descends from the action of T by translations on the right factor of the product $\widetilde{M/T} \times T$.*

REMARK 3.3.2. This remark justifies that item ii) in Definition 3.3.1 above is correctly defined.

The pull-back of the symplectic form ν on M/T, given by Lemma 3.2.1, to the universal cover $\widetilde{M/T}$, by means of the smooth covering map $\psi\colon \widetilde{M/T} \to M/T$, is a $\pi_1^{\mathrm{orb}}(M/T,\, p_0)$-invariant symplectic form on the orbifold universal cover $\widetilde{M/T}$. The symplectic form on the torus T is translation invariant and therefore $\pi_1^{\mathrm{orb}}(M/T,\, p_0)$-invariant. The direct sum of the symplectic form on $\widetilde{M/T}$ and the symplectic form on T is a $\pi_1^{\mathrm{orb}}(M/T,\, p_0)$-invariant (and T-invariant) symplectic form on the Cartesian product $\widetilde{M/T} \times T$, and therefore there exists a unique symplectic form on the associated bundle $\widetilde{M/T} \times_{\pi_1^{\mathrm{orb}}(M/T,\, p_0)} T$ of which the pull-back by the covering map $\widetilde{M/T} \times T \to \widetilde{M/T} \times_{\pi_1^{\mathrm{orb}}(M/T,\, p_0)} T$ is equal to the given symplectic form on $\widetilde{M/T} \times T$.

If T acts freely on M in Definition 3.3.1, the orbit space M/T is a smooth manifold and the universal covering $\widetilde{M/T} \to M/T$ is a principal $\pi_1(M/T,\, p_0)$-bundle over M/T. The homomorphism $\mu\colon \pi_1(M/T,\, p_0) \to T$ gives rise to a representation $[\gamma] \mapsto (t \mapsto \mu([\gamma]) \cdot t)$ of $\pi_1(M/T,\, p_0)$ in the automorphism group of T. The fiber bundle associated to $\widetilde{M/T} \to M/T$ by this representation is the space $\widetilde{M/T} \times_{\pi_1(M/T,\, p_0)} T$, and by similarity we used the term "associated bundle" to refer to it in the more general case of M/T being an orbifold.

REMARK 3.3.3. In the theory of group actions it is more frequent to write the group acting on the left, i.e. to write $T \times_{\pi_1(M/T,\, p_0)} \widetilde{M/T}$ instead of $\widetilde{M/T} \times_{\pi_1(M/T,\, p_0)} T$, while this latter notation would be more common in the theory of fiber bundles, to emphasize T "as a fiber".

EXAMPLE 3.3.4. When $\dim M - \dim T = 2$, M/T is 2-dimensional and it is an exercise to describe locally the monodromy homomorphism μ which appears in part i) of Definition 3.3.1. A small T-invariant open subset of our symplectic manifold looks like $T \times_{T_x} D$, where T is the standard 2-dimensional torus $(\mathbb{R}/\mathbb{Z})^2$, and D is a standard 2-dimensional disk centered at the origin in the complex plane $\mathbb{C}$. Here the quotient M/T is the orbisurface D/T_x. (Recall that we know that T_x is a finite cyclic group, c.f. Lemma 2.2.6). Suppose that T_x has order n. The monodromy homomorphism μ in Definition 3.3.1 part i) is a map from $\pi_1^{\mathrm{orb}} := \pi_1^{\mathrm{orb}}(D/T_x,\, p_0) = T_x = \langle \gamma \rangle$ into T, with γ of order n. If $t \in T$, there exists a homomorphism $f\colon \pi_1^{\mathrm{orb}} \to T$ such that $f(\gamma) = t$ if and only if $t^n = 1$.

If we identify T with $(\mathbb{R}/\mathbb{Z})^2$, we can write $t = (t_1,\, t_2)$. There exists a homomorphism $f\colon \pi_1^{\mathrm{orb}} \to T$ such that $f(\gamma) = (t_1,\, t_2)$ if and only if n divides the order of $(t_1,\, t_2)$, which means that $t_1,\, t_2$ must be rational numbers such that n divides the smallest integer m such that $m\, t_i \in \mathbb{Z}$, $i = 1, 2$. If for example $n = 2$, this condition says that the smallest integer m such that $m\,t_i \in \mathbb{Z}$ must be an even number. In other words, not every element in T can be achieved by the monodromy homomorphism. In fact, all the achievable elements are of finite order, but as we see from this example, more restrictions must take place.

3.4. Model of $(M,\, \sigma)$: proof

We prove that the associated bundle in Definition 3.3.1, which we called "the model of M", is T-equivariantly symplectomorphic to $(M,\, \sigma)$. The main ingredient of the proof is the existence of the flat connection for $\pi\colon M \to M/T$ in Proposition 2.4.1.

We start with the observation that the universal cover $\widetilde{M/T}$ is a smooth manifold and the orbit space M/T is a good orbifold, c.f. Definition 9.1.3.

Lemma 3.4.1. *Let $(M,\, \sigma)$ be a compact connected symplectic manifold equipped with an effective symplectic action of a torus T for which at least one, and hence every T-orbit is a $\dim T$-dimensional symplectic submanifold of $(M,\, \sigma)$. Then the orbifold universal cover $\widetilde{M/T}$ is a smooth manifold and the orbit space M/T is a good orbifold. Moreover, if $\dim T = \dim M - 2$, the orbit space M/T is a very good orbifold.*

Proof. Let $\psi\colon \widetilde{M/T} \to M/T$ be the universal cover of the orbit space M/T based at the regular point $p_0 = \pi(x_0) \in M/T$. Recall from Proposition 2.4.1 the connection Ω for the principal T-orbibundle $\pi\colon M \to M/T$ projection mapping, whose elements are the symplectic orthogonals to the tangent spaces to the T-orbits of $(M,\, \sigma)$. By Corollary 3.1.1, if the mapping $i_x\colon \mathcal{I}_x \to M$ is the inclusion mapping of the integral manifold $\mathcal{I}_x$ through x to the distribution Ω, then the composite map $\pi \circ i_x\colon \mathcal{I}_x \to M/T$ is an orbifold covering mapping, for each $x \in M$, in which the total space $\mathcal{I}_x$ is a smooth manifold. Because the covering map $\psi\colon \widetilde{M/T} \to M/T$ is universal, there exists an orbifold covering $r\colon \widetilde{M/T} \to \mathcal{I}_x$ such that $\pi \circ i_x \circ r = \psi$, and in particular the orbifold universal cover $\widetilde{M/T}$ is a smooth manifold, since an orbifold covering of a smooth manifold must be a smooth manifold itself. Since the orbit space M/T is obtained as a quotient of $\widetilde{M/T}$ by the discrete group $\pi_1^{\mathrm{orb}}(M/T,\, p_0)$, by definition $(M/T,\, \mathcal{A})$ is a good orbifold. Now, it is well-known [**6**, Sec. 2.1.2] that in dimension 2, an orbifold is good if and only if it is very good. $\square$

The following is a particular case of Lemma 9.2.6.

Lemma 3.4.2. *Let T be a torus and let $(M,\, \sigma)$ be a compact connected symplectic manifold endowed with an effective symplectic action of T for which at least one, and hence every T-orbit is a $\dim T$-dimensional symplectic submanifold of $(M,\, \sigma)$. Let $p_0 = \pi(x_0) \in M/T$ be a regular point. Then for any loop $\gamma\colon [0,\, 1] \to M/T$ such that $\gamma(0) = p_0$ there exists a* *unique horizontal lift $\lambda_\gamma\colon [0,1] \to M$ with respect to the connection Ω for $\pi\colon M \to M/T$ in Proposition 2.4.1, such that $\lambda_\gamma(0) = x_0$.*

The following is the main result of Chapter 2.

THEOREM 3.4.3. *Let (M, σ) be a compact connected symplectic manifold equipped with an effective symplectic action of a torus T, for which at least one, and hence every T-orbit is a $\dim T$-dimensional symplectic submanifold of (M, σ). Then (M, σ) is T-equivariantly symplectomorphic to its T-equivariant symplectic model based at any regular point $p_0 \in M/T$, c.f. Definition 3.3.1.*

PROOF. Let $\psi\colon \widetilde{M/T} \to M/T$ be the universal cover of orbit space M/T based at the point $p_0 = \pi(x_0) \in M/T$. Recall from Proposition 2.4.1 the connection Ω for the T-orbibundle $\pi\colon M \to M/T$ projection mapping given by the symplectic orthogonals to the tangent spaces to the T-orbits of (M, σ). By Lemma 3.4.1, $\widetilde{M/T}$ is a smooth manifold. The mapping $\pi_1^{\mathrm{orb}}(M/T, p_0) \times \widetilde{M/T} \to \widetilde{M/T}$ given by $([\lambda], [\gamma]) \mapsto [\gamma\, \lambda]$ is a smooth action of the orbifold fundamental group $\pi_1^{\mathrm{orb}}(M/T, p_0)$ on the orbifold universal cover $\widetilde{M/T}$, which is transitive on each fiber $\widetilde{M/T}_p$ of $\psi\colon \widetilde{M/T} \to M/T$. By Lemma 3.4.2, for any loop $\gamma\colon [0,1] \to M/T$ in the orbifold M/T such that $\gamma(0) = p_0$, denote by $\lambda_\gamma\colon [0,1] \to M$ its unique horizontal lift with respect to the connection Ω for $\pi\colon M \to M/T$ such that $\lambda_\gamma(0) = x_0$, where by horizontal we mean that $\mathrm{d}\,\lambda_\gamma(t)/\,\mathrm{d}\,t \in \Omega_{\lambda_{\gamma(t)}}$ for every $t \in [0,1]$. Proposition 2.4.1 says that Ω is an orbifold flat connection, which means that $\lambda_\gamma(1) = \lambda_\delta(1)$ if δ is homotopy equivalent to γ in the space of all orbifold paths in the orbit space M/T which start at p_0 and end at the given end point $p = \gamma(1)$. The gives existence of a unique group homomorphism $\mu\colon \pi_1^{\mathrm{orb}}(M/T, p_0) \to T$ such that $\lambda_\gamma(1) = \mu([\gamma]) \cdot x_0$. The homomorphism μ does not depend on the choice of the base point $x_0 \in M$. For any homotopy class $[\gamma] \in \widetilde{M/T}$ and $t \in T$, define the element

$$\Phi([\gamma], t) := t \cdot \lambda_\gamma(1) \in M. \tag{3.4.1}$$

The assignment $([\gamma], t) \mapsto \Phi([\gamma], t)$ defines a smooth covering $\Phi\colon \widetilde{M/T} \times T \to M$ between smooth manifolds. Let $[\delta] \in \pi_1^{\mathrm{orb}}(M/T, p_0)$ act on the product $\widetilde{M/T} \times T$ by sending the pair $([\gamma], t)$ to the pair $([\gamma\, \delta^{-1}], \mu([\delta])\, t)$. One can show that this action is free, and hence the associated bundle $\widetilde{M/T} \times_{\pi_1^{\mathrm{orb}}(M/T,\, p_0)} T$ is a smooth manifold.

Next we show this freeness property. Let $\mathcal{I}_x$ be the integral manifold defined in Lemma 3.1.1. Let $q_0 \in \mathcal{I}_x/S$, a regular point of $\mathcal{I}_x/S$. Choose $\widetilde{q}_0 \in \mathcal{I}_x$ over q_0; this means that $q_0 = S \cdot \widetilde{q}_0$ and $T_{\widetilde{q}_0} = \{1\}$. Let $\widetilde{\mathcal{I}_x}$ be the universal cover of $\mathcal{I}_x$ based at $\widetilde{q}_0$. Consider the following diagram, where $i_x\colon \mathcal{I}_x \to M$ is the inclusion map, S is the subgroup of T, which does not depend on the choice of x, defined by

$$S = \{t \in T \,|\, t \cdot \mathcal{I}_x = \mathcal{I}_x\} = \{t \in T \,|\, (t \cdot \mathcal{I}_x) \cap \mathcal{I}_x \neq \emptyset\} \subset T,$$

and $\pi_{\mathcal{I}_x} \colon \mathcal{I}_x \to \mathcal{I}_x/S$ is the canonical projection:

$$\begin{tikzcd}
\widetilde{\mathcal{I}_x} \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & & \widetilde{M/T} \arrow[ll, "\simeq"'] \arrow[ddl] \\
\mathcal{I}_x \arrow[r, "i_x"] \arrow[d, "\pi_{\mathcal{I}_x}"] \arrow[dr, phantom, "\circlearrowleft"] & M \arrow[d, "\pi"] & \\
\mathcal{I}_x/S \arrow[r, "f_x"] & M/T &
\end{tikzcd} \tag{3.4.2}$$

Here the diagonal arrow is the orbifold universal covering as constructed in Section 9.2. The space $\mathcal{I}_x/S$ has an orbifold structure inherited from the manifold structure of $\mathcal{I}_x$ by the proper action of S. The bottom map f_x is the unique map which makes the bottom square of the diagram commutative, or equivalently, it is the map induced by the inclusions $\mathcal{I}_x \hookrightarrow M$ and $S \hookrightarrow T$. By construction f_x is an orbifold diffeomorphism. Since by Lemma 3.1.1 the composite map $\pi \circ i_x$ is an orbifold covering map and the diagram is commutative, the projection $\pi_{\mathcal{I}_x}$ is an orbifold covering map (which is also immediate from its definition, more so than in Lemma 3.1.1).

The composite map of the two vertical left arrows with the bottom horizontal arrow in the diagram (3.4.2) is a composite of a covering, an orbifold covering, and an orbifold diffeomorphism, and hence it is itself an orbifold covering of M/T. It follows from the universality property of the covering $\widetilde{M/T} \to M/T$ that there is a covering map $\widetilde{M/T} \to \widetilde{\mathcal{I}_x}$. Because $\widetilde{\mathcal{I}_x}$ is smooth this orbifold covering map is an ordinary covering map, and because $\widetilde{\mathcal{I}_x}$ is simply connected, it is a diffeomorphism. It follows that $\mathcal{I}_x/S$ is diffeomorphic to $\widetilde{\mathcal{I}_x}/\pi_1^{\mathrm{orb}}(\mathcal{I}_x/S,\, q_0)$ and that the induced map $(\pi_{\mathcal{I}_x})_* \colon \pi_1(\mathcal{I}_x,\, \widetilde{q}_0) \to \pi_1^{\mathrm{orb}}(\mathcal{I}_x/S,\, q_0)$ is an injective homomorphism; indeed suppose that the homotopy class of $\pi_{\mathcal{I}_x} \circ \delta$ is trivial in $\pi_1^{\mathrm{orb}}(\mathcal{I}_x/S,\, q_0)$. This is equivalent to $\pi_{\mathcal{I}_x} \circ \delta$ being contractible in $\mathcal{I}_x/S$ by means of an orbifold homotopy of loops, which in turn is equivalent to $\mathcal{I}_x/S$ lifting uniquely in arbitrary orbifold charts. But since the open neighborhoods of points in $\mathcal{I}_x$ act as orbifold charts, this defines a homotopy of δ to a point.

Consider the composite homomorphism μ' defined by the following diagram, where recall μ was the monodromy homomorphism previously defined:

$$\begin{tikzcd}
\pi_1^{\mathrm{orb}}(\mathcal{I}_x/S,\, q_0) \arrow[d, "(f_x)_*"] \arrow[dr, "\mu'"] & \\
\pi_1^{\mathrm{orb}}(M/T,\, p_0) \arrow[r, "\mu"] & S \subset T
\end{tikzcd}$$

where note the vertical arrow is a canonical identification. We claim that

$$\ker(\mu') = (\pi_{\mathcal{I}_x})_*(\pi_1(\mathcal{I}_x,\, \widetilde{q}_0)), \tag{3.4.3}$$

which in particular implies that the group $(\pi_{\mathcal{I}_x})_*(\pi_1(\mathcal{I}_x,\, \widetilde{q}_0))$ is a normal subgroup of $\pi_1^{\mathrm{orb}}(\mathcal{I}_x/S,\, q_0)$, and we have a commutative diagram

$$\begin{tikzcd}
\pi_1^{\mathrm{orb}}(\mathcal{I}_x/S,\, q_0) \arrow[r] \arrow[d, "(f_x)_*"] \arrow[dr, "\mu'"] & \pi_1^{\mathrm{orb}}(\mathcal{I}_x/S,\, q_0)/(\pi_{\mathcal{I}_x})_*(\pi_1(\mathcal{I}_x,\, \widetilde{q}_0)) \arrow[d, "\simeq"] \\
\pi_1^{\mathrm{orb}}(M/T,\, p_0) \arrow[r, "\mu"] & S \subset T
\end{tikzcd}$$

with the top arrow being the quotient map. Indeed, take an orbifold loop δ in $\mathcal{I}_x/S$ based at q_0. This implies that there exists a curve γ in $\mathcal{I}_x$ starting at $\widetilde{q}_0$ and such that $\delta = \pi_{\mathcal{I}_x} \circ \gamma$. We have $\delta(1) = \delta(0)$; this means that $\pi_{\mathcal{I}_x}(\gamma(1)) = \pi_{\mathcal{I}_x}(\gamma(0))$, which is equivalent to the existence of $t \in T$ such that $\gamma(1) = t\cdot\gamma(0) = t\cdot\widetilde{q}_0$. This t is unique, and by definition $t = \mu'([\delta])$. We have that $[\delta] \in \ker(\mu')$ if and only if $t = 1$ if and only if $\gamma(1) = \gamma(0)$ if and only if γ is an ordinary loop in $\mathcal{I}_x$ if and only if $[\delta] \in (\pi_{\mathcal{I}_x})_*(\pi_1(\mathcal{I}_x,\, \widetilde{q}_0))$. Hence (3.4.3). Since $\pi_1(\mathcal{I}_x)$ acts freely on $\widetilde{\mathcal{I}_x}$ because $\mathcal{I}_x$ is smooth, it follows that the kernel of μ' acts freely on $\widetilde{\mathcal{I}_x}$. Because f_x is an orbifold diffeomorphism, this implies that the kernel of μ acts freely on the simply connected smooth manifold $\widetilde{M/T}$. Therefore the action of $\pi_1^{\mathrm{orb}}(M/T,\, p_0)$ on the product $\widetilde{M/T} \times T$ is free, as claimed.

The mapping Φ induces a diffeomorphism ϕ from the associated bundle $\widetilde{M/T} \times_{\pi_1^{\mathrm{orb}}(M/T,\, p_0)} T$ onto M. Indeed, ϕ is onto because Φ is onto, and $\Phi([\gamma],\, t) = \Phi([\gamma'],\, t')$ if and only if $t \cdot \lambda_\gamma(1) = t' \cdot \lambda_{\gamma'}(1)$, if and only if $\lambda_\gamma(1) = (t^{-1}\, t') \cdot \lambda_{\gamma'}(1)$, if and only if $t^{-1}\, t' = \mu([\delta])$, in which δ is equal to the loop starting and ending at p_0, which is obtained by first doing the path γ and then going back by means of the path γ'^{-1}. Since Φ is a smooth covering map, the mapping ϕ is a local diffeomorphism, and we have just proved that it is bijective, so ϕ must be a diffeomorphism. By definition, ϕ intertwines the action of T by translations on the right factor of the associated bundle $\widetilde{M/T} \times_{\pi_1^{\mathrm{orb}}(M/T,\, p_0)} T$ with the action of T on M. Recall that the symplectic form on $\widetilde{M/T} \times_{\pi_1^{\mathrm{orb}}(M/T,\, p_0)} T$ is the unique symplectic form of which the pullback $\widetilde{M/T} \times T \to \widetilde{M/T} \times_{\pi_1^{\mathrm{orb}}(M/T,\, p_0)} T$ is equal to the product form on $\widetilde{M/T} \times T$, where the symplectic form on $\widetilde{M/T}$ is given by Definition 3.3.1 part ii). It follows from the definition of the symplectic form on $\widetilde{M/T} \times_{\pi_1^{\mathrm{orb}}(M/T,\, p_0)} T$ and Corollary 2.5.1 that the T-equivariant diffeomorphism ϕ from $\widetilde{M/T} \times_{\pi_1^{\mathrm{orb}}(M/T,\, p_0)} T$ onto M pulls back the symplectic form σ on M to the just obtained symplectic form on $\widetilde{M/T} \times_{\pi_1^{\mathrm{orb}}(M/T,\, p_0)} T$. $\square$

Remark 3.4.4. In the proof of Theorem 3.4.3 we have provided an alternative description of the orbifold M/T as a quotient $\mathcal{I}_x/S$, i.e. M/T

and $\mathcal{I}_x/S$ are canonically isomorphic as orbifolds. Had we introduced this description from the beginning, the proof of Lemma 3.1.1 would have been immediate, for example. On the other hand the definition of the model of (M, σ) involves less notation with our description. Other than this, both view-points are equivalent.

REMARK 3.4.5. We assume the terminology of Definition 3.3.1 and Theorem 3.4.3. In this remark we describe a covering isomorphic to the orbifold covering $\pi \circ i_x$ of Lemma 3.1.1, which will be of use in Theorem 4.3.2. Additonally, the construction of this new covering makes transparent the relation between the universal cover $\widetilde{M/T} \to M/T$ and the covering $\mathcal{I}_x \to M/T$. Indeed, recall the distribution Ω of the symplectic orthogonal complements of the tangent spaces of the T-orbits given in Proposition 2.4.1, and the principal T-orbibundle $\pi : M \to M/T$, for which Ω is a T-invariant flat connection. As in the proof of Theorem 3.4.3, if $x \in M$ then each smooth curve δ in M/T which starts at $\pi(x)$ has a unique horizontal lift γ which starts at x. The endpoints of such lifts form the injectively immersed manifold $\mathcal{I}_x$, c.f. Remark 3.1.2. As in the proof of Theorem 3.4.3 , if we keep the endpoints of δ fixed, then the endpoint of γ only depends on the homotopy class of δ. As explained prior to Definition 3.3.1, the homotopy classes of δ's in M/T with fixed endpoints are by definition the elements of the universal covering space $\widetilde{M/T}$ of M/T. The corresponding endpoints of the γ's exhibit the integral manifold $\mathcal{I}_x$ as the image of an immersion from $\widetilde{M/T}$ into M, but this immersion is not necessarily injective. Recall that the monodromy homomorphism μ of Ω tells what the endpoint of γ is when δ is a loop. By replacing μ by the induced injective homomorphism μ' from $\pi_1^{\mathrm{orb}}(M/T, \pi(x))/\ker\mu$ to T, we get an injective immersion. This procedure is equivalent to replacing the universal covering $\widetilde{M/T}$ by the covering $\widetilde{M/T}/\ker\mu$ with fiber $\pi_1^{\mathrm{orb}}(M/T, \pi(x))/\ker\mu$. So $\widetilde{M/T}/\ker\mu$ is injectively immersed in M by a map whose image is $\mathcal{I}_x$. It follows that the coverings $\pi \circ i_x \colon \mathcal{I}_x \to M/T$ and $\widetilde{M/T}/\ker\mu \to M/T$ are isomorphic coverings of M/T, of which the total space is a smooth manifold. We use this new covering $\widetilde{M/T}/\ker\mu \to M/T$ in order to construct an alternative model of (M, σ) to the one given in Theorem 3.4.3, c.f. Theorem 4.3.2. Note: in the statement of Theorem 4.3.2, $N := \widetilde{M/T}/\ker\mu$.

EXAMPLE 3.4.6. Assume the terminology of Definition 3.3.1. Assume moreover that $\dim M - \dim T = 2$ and that the action of T on M is free, so the quotient space M/T is a compact, connected, smooth surface and it is classified by its genus. If the genus is zero, then the orbit space M/T is a sphere, which is simply connected, and the orbibundle $\pi : \mathcal{I}_x \to M/T$ is a diffeomorphism, c.f. Lemma 3.1.1. In such case M is the Cartesian product of a sphere with a torus. If M/T has genus 1, then the orbit space M/T is a two-dimensional torus, with fundamental group at any point isomorphic to the free abelian group on two generators t_1, t_2, and the monodromy homomorphism is determined by the images t_1 and t_2 of the generators $(1, 0)$ and $(0, 1)$ of $\mathbb{Z}^2$. $\mathcal{I}_x$ is compact if and only if $\mathcal{I}_x$ is a closed subset of M if and only if t_1 and t_2 generate a closed subgroup of T if and only if t_1 and t_2 generate a finite subgroup of T. $\mathcal{I}_x$ is dense in M if and only if the subgroup of T generated by t_1 and t_2 is dense in T. If the genus of M/T is strictly positive, then $\mathcal{I}_x$ is compact if and only if the monodromy elements form a finite subgroup of T, which is a very particular situation (since the $2g$ generators of the monodromy subgroup of T can be chosen arbitrarily, this is very rare: even one element of T usually generates a dense subgroup of T).

CHAPTER 4

Global model up to equivariant diffeomorphisms

Throughout this chapter $(M,\, \sigma)$ is a compact and connected symplectic manifold and T is a torus which acts effectively on $(M,\, \sigma)$ by means of symplectomorphisms, and such that at least one T-orbit is a $\dim T$-dimensional symplectic submanifold of $(M,\, \sigma)$. In the first two sections we shall moreover assume that T acts freely and provide a model of $(M,\, \sigma)$ up to T-equivariant diffeomorphisms.

4.1. Generalization of Kahn's theorem

In [**25**, Cor. 1.4] P. Kahn states that if a compact connected 4-manifold M admits a free action of a 2-torus T such that the T-orbits are 2-dimensional symplectic submanifolds, then M splits as a product $M/T \times T$, the following being the statement in [**25**].

THEOREM 4.1.1 (P. Kahn, [**25**], Cor. 1.4). *Let $(M,\, \sigma)$ be a compact connected symplectic 4-dimensional manifold. Suppose that M admits a free action of a 2-dimensional torus T for which the T-orbits are 2-dimensional symplectic submanifolds of $(M,\, \sigma)$. Then there exists a T-equivariant diffeomorphism between M and the product $M/T \times T$, where T is acting by translations on the right factor of $M/T \times T$.*

Next we generalize this result of Kahn's to the case when the torus and the manifold are of arbitrary dimension.

COROLLARY 4.1.2. *Let $(M,\, \sigma)$ be a compact connected $2n$-dimensional symplectic manifold equipped with a free symplectic action of a $(2n-2)$--dimensional torus T such that at least one, and hence every T-orbit is a $(2n-2)$-dimensional symplectic submanifold of $(M,\, \sigma)$. Then M is T--equivariantly diffeomorphic to the product $M/T \times T$, where $M/T \times T$ is equipped with the action of T on the right factor of $M/T \times T$ by translations.*

The proof of Corollary 4.1.2 relies on the forthcoming Theorem 4.2.1, so we shall prove it after we state and prove the theorem.

4.2. Smooth equivariant splittings

We give a characterization of the existence of T-equivariant splittings of M as a Cartesian product $M/T \times T$, up to T-equivariant diffeomorphisms c.f. Theorem 4.2.1.

Recall that if X is a smooth manifold, $\mathrm{H}_1(X,\,\mathbb{Z})_{\mathrm{T}}$ denotes the set of torsion elements of $\mathrm{H}_1(X,\,\mathbb{Z})$ i.e. the set of $[\gamma] \in \mathrm{H}_1(X,\,\mathbb{Z})$ such that there exists a strictly positive integer k satisfying $k\,[\gamma] = 0$. Let $\mu_{\mathrm{h}}^{\mathrm{T}}$ denote the homomorphism given by restricting the homomorphism μ_{h} to $\mathrm{H}_1(M/T,\,\mathbb{Z})_{\mathrm{T}}$. Recall that by Proposition 2.3.4, M is a compact, connected $(\dim M - \dim T)$-dimensional orbifold, and that if the action of torus T on M is free, the local groups are trivial, and hence the orbit space M/T is a $(\dim M - \dim T)$-dimensional smooth manifold, c.f. Remark 2.3.6.

Theorem 4.2.1. *Let $(M,\,\sigma)$ be a compact connected symplectic manifold equipped with a free symplectic action of a torus T for which at least one, and hence every of its T-orbits is a $\dim T$-dimensional symplectic submanifold of $(M,\,\sigma)$, and let $\mu_{\mathrm{h}} \colon\, \mathrm{H}_1(M/T,\,\mathbb{Z}) \to T$ be the homomorphism induced on homology via the Hurewicz map by the monodromy homomorphism μ from $\pi_1(M/T,\,p_0)$ into T with respect to the connection on M given by the symplectic orthogonal complements to the tangent spaces to the T-orbits, for the T-bundle projection map $\pi \colon M \to M/T$ onto the orbit space M/T (c.f. expression (4.3.1)). Then M is T-equivariantly diffeomorphic to the Cartesian product $M/T \times T$ equipped with the action of T by translations on the right factor of $M/T \times T$, if and only if the torsion part $\mu_{\mathrm{h}}^{\mathrm{T}}$ of the homomorphism μ_{h} is trivial, i.e. μ_{h} satisfies that $\mu_{\mathrm{h}}([\gamma]) = 1$ for every $[\gamma] \in \mathrm{H}_1(M/T,\,\mathbb{Z})$ of finite order.*

Proof. By means of the same argument that we used in the proof of Theorem 3.4.3, using any T-invariant flat connection $\mathcal{D}$ for π instead of Ω, we may define a mapping

$$\phi^{\mathcal{D}} \colon \widetilde{M/T} \times_{\pi_1(M/T,\,p_0)} T \to M. \tag{4.2.1}$$

The mapping $\phi^{\mathcal{D}}$, which is induced by $\widetilde{(3.4.1)}$, is a T-equivariant diffeomorphism between the smooth manifolds $\widetilde{M/T} \times_{\pi_1(M/T,\,p_0)} T$ and M, where $\widetilde{M/T}$ is the universal cover of the manifold M/T based at p_0. Since $\widetilde{M/T}$ is a regular covering c.f. Remark 4.2.3, $\widetilde{M/T}/\pi_1(M/T,\,p_0) = M/T$ and the monodromy homomorphism mapping $\mu^{\mathcal{D}} \colon \pi_1(M/T,\,p_0) \to T$ of $\mathcal{D}$ is trivial if and only if $\widetilde{M/T} \times_{\pi_1(M/T,\,p_0)} T = M/T \times T$. Hence the smooth manifold M is T-equivariantly diffeomorphic to the Cartesian product $M/T \times T$ if and only if there exists a T-invariant flat connection $\mathcal{D}$ for the principal T-bundle $\pi \colon M \to M/T$ the monodromy homomorphism of which $\mu^{\mathcal{D}} \colon \pi_1(M/T,\,p_0) \to T$ is trivial.

If $\mu_{\mathrm{h}}^{\mathrm{T}} = 0$, there exists a homomorphism $\widehat{\mu_{\mathrm{h}}} \colon\, \mathrm{H}_1(M/T,\,\mathbb{Z}) \to \mathfrak{t}$ such that $\exp \circ \widehat{\mu_{\mathrm{h}}} = \mu_{\mathrm{h}}$, which by viewing $\widehat{\mu_{\mathrm{h}}}$ as an element $[\beta] \in \mathrm{H}^1_{\text{de Rham}}(M/T) \otimes \mathfrak{t}$, can be rewritten as

$$\mu_{\mathrm{h}}([\gamma]) = \exp \int_{\gamma} \beta. \tag{4.2.2}$$

The T-invariant connection Ω of symplectic orthogonal complements to the tangent spaces to the T-orbits defines a T-invariant connection one-form $\theta \in \Omega^1(M) \otimes \mathfrak{t}$ for the canonical projection mapping $\pi\colon M \to M/T$. By Proposition 2.4.1, the connection Ω is flat, and since the torus T is an abelian group, θ is a closed form. The one-form $\Theta := \theta - \pi^*\beta \in \Omega^1(M) \otimes \mathfrak{t}$ is a $\mathfrak{t}$-valued connection one-form on M, and therefore there exists a T-invariant connection $\mathcal{D}$ on M, such that $\Theta = \widehat{\theta}^{\mathcal{D}}$. Since β is closed, and Ω is flat, the connection one-form $\widehat{\theta}^{\mathcal{D}}$ is flat, and hence the connection $\mathcal{D}$ on M is flat. This means that $\mathcal{D}$ has a monodromy homomorphism $\mu^{\mathcal{D}}\colon \pi_1(M/T,\, p_0) \to T$, and a corresponding monodromy homomorphism on homology $\mu^{\mathcal{D}}_{\mathrm{h}}\colon\ \mathrm{H}_1(M/T,\, \mathbb{Z}) \to \mathfrak{t}$, and for any closed curve γ in the orbit space M/T,

$$\mu^{\mathcal{D}}_{\mathrm{h}}([\gamma]) = \mu_{\mathrm{h}}([\gamma])\ \exp \int_\gamma -\beta = \mu_{\mathrm{h}}([\gamma])\ \ (\exp \int_\gamma \beta)^{-1} = 1,$$

where the second equality follows from the fact that

$$\exp \int_\gamma -\beta = (\exp \int_\gamma \beta)^{-1},$$

and the third equality follows from (4.2.2). We have proven that $\mathcal{D}$ is a flat connection for the orbibundle $\pi\colon M \to M/T$ whose monodromy homomorphism on homology $\mu^{\mathcal{D}}_{\mathrm{h}}$ is trivial, and hence so it is $\mu^{\mathcal{D}}$. Conversely, if M is T-equivariantly diffeomorphic to the Cartesian product $M/T \times T$, then there exists a T-invariant flat connection $\mathcal{D}$ for the orbibundle $\pi\colon M \to M/T$ such that the monodromy homomorphism for π with respect to $\mathcal{D}$, $\mu^{\mathcal{D}}\colon \pi_1(M/T,\, p_0) \to T$ is trivial. On the other hand, if $[\gamma] \in \mathrm{H}_1(M/T,\, \mathbb{Z})_{\mathrm{T}}$, there exists a strictly positive integer k with $k\,[\gamma] = 0$ and $0 = \langle k\,[\gamma],\, [\alpha]\rangle = k\,\langle[\gamma],\, [\alpha]\rangle$, which means that $\langle[\gamma],\, [\alpha]\rangle = 0$ for all $[\gamma] \in \mathrm{H}_1(M/T,\, \mathbb{Z})_{\mathrm{T}}$, and hence that $(\mu^{\mathcal{D}}_{\mathrm{h}})^{\mathrm{T}} = \mu^{\mathrm{T}}_{\mathrm{h}}$. Since the homomorphism $\mu^{\mathcal{D}}$ is trivial, $\mu^{\mathrm{T}}_{\mathrm{h}}$ is trivial. □

Corollary 4.1.2 follows immediately. Indeed, if $\dim(M/T) = 2$, the quotient space M/T is a compact, connected, smooth, orientable surface. Therefore, by the classification theorem for surfaces, $\mathrm{H}_1(M/T,\, \mathbb{Z})$ is isomorphic to $\mathbb{Z}^{2g}$, where g is the topological genus of M/T. Therefore $\mathrm{H}_1(M/T,\, \mathbb{Z})$ does not have torsion elements, and by Theorem 4.2.1, M is T-equivariantly diffeomorphic to the cartesian product $M/T \times T$.

Remark 4.2.2. Under the assumptions of Theorem 4.2.1 on our symplectic manifold $(M,\, \sigma)$, if the dimension of the orbit space M/T is strictly greater than 2, then it may happen that the first integral orbifold homology group $\mathrm{H}_1(M/T,\, \mathbb{Z})$ has no torsion and therefore M is T-equivariantly diffeomorphic to the Cartesian product $M/T \times T$. However, already in the case that M/T is 4-dimensional there are examples of symplectic manifolds M/T of which the integral homology group $\mathrm{H}_1(M/T,\, \mathbb{Z})$ has non-trivial torsion. For instance, in [**12**, Sec. 8] we computed the fundamental group and the first integral homology group of the whole manifold, and this computation shows that even in dimension 4 there are examples X where $\mathrm{H}_1(X,\,\mathbb{Z})$ is isomorphic to $\mathbb{Z}^3 \times (\mathbb{Z}/k\,\mathbb{Z})$, where k can be any positive integer (see also [**34**] for related examples). Taking such manifolds as the base space M/T, and a monodromy homomorphism which is non-trivial on the torsion subgroup $\mathbb{Z}/k\,\mathbb{Z}$, we arrive at an example where M is not T-equivariantly diffeomorphic to the Cartesian product $M/T \times T$.

Remark 4.2.3. Under the assumptions of Theorem 4.2.1, since $\widetilde{M/T}$ is a regular covering of M/T, we have a diffeomorphism $\widetilde{M/T}/\pi_1(M/T,\,p_0) \simeq M/T$ naturally, or in other words, the symbol $\simeq$ may be taken to be an equality. Hence the monodromy homomorphism $\mu^{\mathcal{D}}\colon \pi_1(M/T,\,p_0) \to T$ of a T-invariant flat connection $\mathcal{D}$ is trivial if and only if $\widetilde{M/T} \times_{\pi_1(M/T,\,p_0)} T = M/T \times T$. Strictly speaking, this is not an equality, it is a T-equivariant diffeomorphism

$$\widetilde{M/T} \times_{\pi_1(M/T,\,p_0)} T \to \widetilde{M/T}/\pi_1(M/T,\,p_0) \times T \to M/T \times T,$$

where the first arrow is the identity map, and the second arrow is the identity on the T component of the Cartesian product $(\widetilde{M/T}/\pi_1(M/T,\,p_0)) \times T$, while on the first component it is given by the map $[\,[\gamma]\,]_{\pi_1(M/T,\,p_0)} \mapsto \gamma(1)$. If the action of T on M is not free, the same argument works by replacing the fundamental group of M/T at p_0, by the corresponding orbifold fundamental group.

4.3. Alternative model

Next we present a model for $(M,\,\sigma)$, up to T-equivariant symplectomorphisms, which does not involve the universal cover of M/T, but rather a smaller cover of M/T.

Definition 4.3.1. *Let $(M,\,\sigma)$ be a compact connected symplectic manifold equipped with an effective symplectic action of a torus T for which at least one, and hence every T-orbit is a* $\dim T$*-dimensional symplectic submanifold of $(M,\,\sigma)$, and let h_1 from $\pi_1^{\mathrm{orb}}(M/T,\,p_0)$ to $\mathrm{H}_1^{\mathrm{orb}}(M/T,\,\mathbb{Z})$ be the orbifold Hurewicz mapping (c.f. Section 9.4). There exists a unique homomorphism $\mathrm{H}_1^{\mathrm{orb}}(M/T,\,\mathbb{Z}) \to T$, which we call μ_{h}, such that*

$$\mu = \mu_{\mathrm{h}} \circ h_1, \tag{4.3.1}$$

where $\mu\colon \pi_1^{\mathrm{orb}}(M/T,\,p_0) \to T$ is the monodromy homomorphism of the connection $\Omega = \{\Omega_x = (\mathrm{T}_x(T \cdot x))^{\sigma_x}\}$ of symplectic orthogonal complements to the tangent spaces to the T-orbits in M (c.f. Proposition 2.4.1). The homomorphism μ_{h} is independent of the choice of base point p_0.

See Remark 3.4.5 for a description of the ingredients involved in the following theorem.

THEOREM 4.3.2. *Let (M,σ) be a compact connected symplectic manifold equipped with an effective symplectic action of a torus T for which at least one, and hence every T-orbit is a $\dim T$-dimensional symplectic submanifold of (M,σ). Then there exist a $(\dim M - \dim T)$-dimensional symplectic manifold (N,σ^N), a commutative group Δ which acts properly on N with finite stabilizers and such that N/Δ is compact, and a group monomorphism $\mu'_{\mathrm{h}}\colon \Delta \to T$, such that the symplectic T-manifold (M,σ) is T-equivariantly symplectomorphic to $N \times_\Delta T$, where Δ acts on $N \times T$ by the diagonal action $x\,(y,\,t) = (x \star y^{-1},\, \mu'_{\mathrm{h}}(x)\cdot t)$, and where $\star\colon \Delta \times N \to N$ denotes the action of Δ on N. $N \times_\Delta T$ is equipped with the action of T by translations which descends from the action of T by translations on the right factor of the product $N \times T$, and the symplectic form induced on the quotient by the product symplectic form $\sigma^N \oplus \sigma^T$ on the product $N \times T$.*

Here σ^T is the unique translation invariant symplectic form on T induced by the antisymmetric bilinear form $\sigma^{\mathfrak{t}}$, where $\sigma_x(X_M(x),\,Y_M(x)) = \sigma^{\mathfrak{t}}(X,Y)$ for every $X,Y \in \mathfrak{t}$, and every $x \in M$, and σ^N is inherited from the symplectic form ν on the orbit space M/T, c.f. Lemma 3.2.1 by means of the covering map $N \to M/T$, c.f. Remark 4.3.3. The structure of smooth manifold for N is inherited from the smooth manifold structure of the orbifold universal cover $\widetilde{M/T}$, since N is defined as the quotient $\widetilde{M/T}/K$ where K is the kernel of the monodromy homomorphism $\mu\colon \pi_1^{\mathrm{orb}}(M/T,\,p_0) \to T$. Moreover, N is a regular covering of the orbifold M/T with covering group Δ.

PROOF. We proved in Theorem 3.4.3 that for any element $[\gamma] \in \widetilde{M/T}$ and $t \in T$, the mapping defined by $\Phi([\gamma],\,t) := t \cdot \lambda_\gamma(1) \in M$, induces a induces a T-equivariant symplectomorphism ϕ from the associated bundle $\widetilde{M/T} \times_{\pi_1^{\mathrm{orb}}(M/T,\,p_0)} T$ onto the symplectic T-manifold (M,σ). See Definition 3.3.1 for the construction of the symplectic form and T-action on this associated bundle. Let K be the kernel subgroup of the monodromy homomorphism μ from the orbifold fundamental group $\pi_1^{\mathrm{orb}}(M/T,\,p_0)$ into the torus T. The kernel subgroup K is a normal subgroup of $\pi_1^{\mathrm{orb}}(M/T,\,p_0)$ which contains the commutator subgroup C of $\pi_1^{\mathrm{orb}}(M/T,\,p_0)$. There exists a unique group homomorphism $\mu_c\colon \pi_1^{\mathrm{orb}}(M/T,\,p_0)/C \to T$ such that $\mu = \mu_c \circ \chi$, where χ is the quotient homomorphism from the orbifold fundamental group $\pi_1^{\mathrm{orb}}(M/T,\,p_0)$ onto $\pi_1^{\mathrm{orb}}(M/T,\,p_0)/C$. The orbifold Hurewicz map h_1 assigns to a homotopy class of a loop based at p_0, which is also a one-dimensional cycle, the homology class of that cycle. h_1 is a homomorphism from $\pi_1^{\mathrm{orb}}(M/T,\,p_0)$ onto $\mathrm{H}_1^{\mathrm{orb}}(M/T,\,\mathbb{Z})$ with kernel the commutator subgroup C. Therefore the quotient group $K/C \leq \pi_1^{\mathrm{orb}}(M/T,\,p_0)/C$ can be viewed as a subgroup of the first orbifold homology group

$$\mathrm{H}_1^{\mathrm{orb}}(M/T,\,\mathbb{Z}) \simeq \pi_1^{\mathrm{orb}}(M/T,\,p_0)/C,$$

where the symbol $\simeq$ stands for the projection induced by the Hurewicz map h_1 from the orbifold fundamental group $\pi_1(M/T,\,p_0)$ into the first orbifold homology group $\mathrm{H}_1^{\mathrm{orb}}(M/T,\,\mathbb{Z})$. Let $N := \widetilde{M/T}/K$. By Lemma 3.1.1 and Remark 3.4.5, N is a smooth manifold (diffeomorphic to the integral manifold of the distribution Ω, c.f. Proposition 2.4.1). Because the universal cover of M/T is a regular orbifold covering of which the orbifold fundamental group is the covering group, the quotient N is a regular orbifold covering of the orbit space M/T with covering group

$$\Delta := \mathrm{H}_1^{\mathrm{orb}}(M/T,\,\mathbb{Z})/(K/C) \simeq \pi_1^{\mathrm{orb}}(M/T,\,p_0)/C.$$

The group Δ is a commutative group. Let Δ act on the torus T by the mapping

$$(x,\,t) \mapsto \mu_{\mathrm{h}}'(x)\,t, \tag{4.3.2}$$

where the mapping $\mu_{\mathrm{h}}' \colon \Delta \to T$ is the quotient homomorphism induced by μ_{h} and π, where recall that μ_{h} denotes the homomorphism induced on homology from the monodromy μ associated to the connection Ω, c.f. Proposition 2.4.1. The mapping μ_{h}' is injective because the subgroup K equals the kernel of μ. Let Δ act on $N \times T$ by the diagonal action, giving rise to $N \times_\Delta T$. Our construction produces an identification of $\widetilde{M/T} \times_{\pi_1^{\mathrm{orb}}(M/T,\,p_0)} T$ and $N \times_\Delta T$, which intertwines the actions of T by translations on the right T-factor of both spaces. In this way the mapping ϕ induces a diffeomorphism from the associated bundle $N \times_\Delta T$ to M, which intertwines the action of T by translations on the right factor of $N \times_\Delta T$ with the action of T on M. By the same proof as in Theorem 4.2.1, the T-equivariant diffeomorphism ϕ from $N \times_\Delta T$ onto M pulls back the symplectic form on M to the symplectic form on $N \times_\Delta T$ given in Remark 4.3.3. $\square$

Remark 4.3.3. This remark justifies why the symplectic form on the model space defined in Theorem 4.3.2 is correctly defined. Let us assume the terminology used in the statement of Theorem 4.3.2. Recall the distribution $\Omega := \{\Omega_x\}_{x \in M}$ on M of symplectic orthogonal complements to the tangent spaces to the T-orbits, defined by Proposition 2.4.1. The pull-back of the 2-form ν on M/T such that $\pi^*\nu|_{\Omega_x} = \sigma|_{\Omega_x}$ for every $x \in M$, to the smooth manifold N by means of the covering map ϕ (c.f. Lemma 3.2.1), is a Δ-invariant symplectic form on N. The symplectic form on T determined by the antisymmetric bilinear form $\sigma^{\mathfrak{t}}$ given by Lemma 2.1.1 is translation invariant, and therefore Δ-invariant. The direct sum of the symplectic form on N and the symplectic form on T is a Δ-invariant and T-invariant symplectic form on $N \times T$, and therefore there is a unique symplectic form on $N \times_\Delta T$ of which the pull-back by the covering map $N \times T \to N \times_\Delta T$ is equal to the given symplectic form on $N \times T$.

CHAPTER 5

Classification: free case

Throughout this chapter (M, σ) is a compact and connected symplectic manifold and T is a torus which acts freely on (M, σ) by means of symplectomorphisms, and such that at least one T-orbit is a $\dim T$-dimensional symplectic submanifold of (M, σ). Our goal is to use the model for (M, σ) which we constructed in Definition 3.3.1 to provide a classification of (M, σ) when $\dim T = \dim M - 2$, in terms of a collection of invariants.

5.1. Monodromy invariant

We define what we call the free monodromy invariant of (M, σ), ingredient 4) in Definition 5.2.1.

5.1.1. Intersection forms and geometric maps.

Let Σ be a compact connected orientable smooth surface of genus g, where g is a positive integer. Recall the *algebraic intersection number*

$$\cap\colon \mathrm{H}_1(\Sigma, \mathbb{Z}) \otimes \mathrm{H}_1(\Sigma, \mathbb{Z}) \to \mathbb{Z}, \tag{5.1.1}$$

which extends uniquely to the *intersection form*

$$\cap\colon \mathrm{H}_1(\Sigma, \mathbb{R}) \otimes \mathrm{H}_1(\Sigma, \mathbb{R}) \to \mathbb{R}, \tag{5.1.2}$$

which turns $\mathrm{H}_1(\Sigma, \mathbb{R})$ into a symplectic vector space. It is always possible to find, c.f. [**21**, Ex. 2A.2], a so called "symplectic" basis, in the sense of [**41**, Th. 2.3].

Definition 5.1.1. *Let Σ be a compact connected orientable smooth surface of genus g, where g is a positive integer. A collection of elements α_i, β_i, $1 \leq i \leq g$, of* $\mathrm{H}_1(\Sigma, \mathbb{Z}) \subset \mathrm{H}_1(\Sigma, \mathbb{R})$ *such that*

$$\alpha_i \cap \alpha_j = \beta_i \cap \beta_j = 0, \;\; \alpha_i \cap \beta_j = \delta_{ij} \tag{5.1.3}$$

for all i, j with $1 \leq i, j \leq g$ is called a symplectic basis of the group $\mathrm{H}_1(\Sigma, \mathbb{Z})$ *or a* symplectic basis of the symplectic vector space $(\mathrm{H}_1(\Sigma, \mathbb{R}), \cap)$
.

Hence the matrix associated to the antisymmetric bilinear form $\cap$ on the basis α_i, β_i is the block diagonal matrix

$$\mathrm{J}_0 = \begin{pmatrix} 0 & 1 & & & & & \\ -1 & 0 & & & & & \\ & & 0 & 1 & & & \\ & & -1 & 0 & & & \\ & & & & \ddots & & \\ & & & & & 0 & 1 \\ & & & & & -1 & 0 \end{pmatrix}.$$

Notice that the symplectic linear group $\mathrm{Sp}(2g, \mathbb{R}) \subset \mathrm{GL}(2g, \mathbb{R})$ is the group of matrices A such that

$$A \cdot \mathrm{J}_0 \cdot A^{\mathrm{t}} = \mathrm{J}_0, \tag{5.1.4}$$

and therefore it is natural to denote by $\mathrm{Sp}(2g, \mathbb{Z})$ the group of matrices $A \in \mathrm{GL}(2g, \mathbb{Z})$ which satisfy expression (5.1.4).

Remark 5.1.2. The group $\mathrm{Sp}(2g, \mathbb{Z})$ is also called Siegel's modular group, and denoted by Γ_g, see for example J. Birman's article [**5**]. Generators for $\mathrm{Sp}(2g, \mathbb{Z})$ were first determined by L. K. Hua and and I. Reiner [**23**]. Later H. Klingen found a characterization [**29**] for $g \geq 2$ by a finite system of relations. Birman's article [**5**] reduces Klingen's article to a more usable form, in which she explicitly describes the calculations in Klingen's paper, among other results. I thank J. McCarthy for making me aware of J. Birman's article and for pointing me to his article [**38**] which contains a generalization of it.

Remark 5.1.3. Let g be a non-negative integer, and let Σ be a compact, connected, oriented, smooth surface of genus g. If the first homology group $\mathrm{H}_1(\Sigma, \mathbb{Z})$ is identified with $\mathrm{GL}(2g, \mathbb{Z})$ by means of a choice of symplectic basis, the group of automorphisms of $\mathrm{H}_1(\Sigma, \mathbb{Z})$ which preserve the intersection form gets identified with the group $\mathrm{Sp}(2g, \mathbb{Z})$ of matrices in $\mathrm{GL}(2g, \mathbb{Z})$ which satisfy expression (5.1.4). Indeed, let α_i, β_i be a symplectic basis of $\mathrm{H}_1(\Sigma, \mathbb{Z})$ (i.e. whose elements satisfy expression (5.1.3)). The $(2g \times 2g)$-matrix $\mathrm{M}(\alpha_i, \beta_i, f)$ of f with respect to the basis α_i, β_i is an element of the linear group $\mathrm{GL}(2g, \mathbb{Z})$ of $(2g \times 2g)$-invertible matrices with integer coefficients and it is an exercise to check that f preserves the intersection form on $\mathrm{H}_1(\Sigma, \mathbb{R})$ if and only if $\mathrm{M}(\alpha_i, \beta_i, f) \cdot \mathrm{J}_0 \cdot \mathrm{M}(\alpha_i, \beta_i, f)^{\mathrm{t}} = \mathrm{J}_0$.

An orientation preserving diffeomorphism induces an isomorphism in homology which preserves the intersection form. Moreover, the converse also holds, c.f. [**36**, pp. 355-356]. An algebraic proof of this result is given by J. Birman in [**5**, pp. 66–67] as a consequence of [**5**, Thm. 1] (see also the references therein). To be self contained we present a sketch of proof here, following a preliminary draft by B. Farb and D. Margalit [**15**]. Denote by $\mathrm{MCG}(\Sigma)$ the mapping class group of orientation preserving diffeomorphisms of Σ modulo isotopies. After a choice of basis, there is a natural homomorphism

$$\rho\colon \mathrm{MCG}(\Sigma) \to \mathrm{Sp}(2g, \mathbb{Z}). \tag{5.1.5}$$

LEMMA 5.1.4. *Let g be a non-negative integer, and let Σ, Σ' be compact, connected, oriented, smooth surfaces of the same genus g. Then a group isomorphism f from the first homology group $\mathrm{H}_1(\Sigma, \mathbb{Z})$ onto the first homology group $\mathrm{H}_1(\Sigma', \mathbb{Z})$ preserves the intersection form if and only if there exists an orientation preserving diffeomorphism $i\colon \Sigma \to \Sigma'$ such that $f = i_*$. Moreover, if $\{U_k\}_{k=1}^m \subset \Sigma$, $\{U'_k\}_{k=1}^m \subset \Sigma'$ are finite disjoint collections of embedded disks, i can be chosen to map U_k to U'_k for all k, $1 \le k \le m$.*

PROOF. W.l.o.g. assume that $\Sigma = \Sigma'$. It is immediate that an orientation preserving diffeomorphism of Σ induces an intersection form preserving automorphism of $\mathrm{H}_1(\Sigma, \mathbb{Z})$. Let γ be the permutation of $\{1, \dots, 2g\}$ which transposes $2i$ and $2i-1$ for $1 \le i \le 2g$. We define the ij^{th} elementary symplectic matrix by

$$\Sigma^{ij} = \begin{cases} I_{2g} + E_{ij} & \text{if } i = \gamma(j); \\ I_{2g} + E_{ij} - (-1)^{i+j} E_{\gamma(i)\gamma(j)} & \text{otherwise.} \end{cases}$$

where I_{2g} stands for the identity matrix of dimension $2g$ and E_{ij} is the matrix with a 1 in the ij^{th} position and $0's$ elsewhere. It is a classical fact that $\mathrm{Sp}(2g, \mathbb{Z})$ is generated by the matrices Σ^{ij}. To prove the lemma is equivalent to showing that ρ in (5.1.5) is surjective onto $\mathrm{Sp}(2g, \mathbb{Z})$. Let τ_b be the Dehn twist about a simple closed curve b. Then for integer values of k, the image $\rho(\tau_b^k)$ is given by

$$a \mapsto a + k \cdot \cap(a, b)\, b. \tag{5.1.6}$$

We may restrict our attention to a subsurface Σ_0 of Σ of genus 1 in the case of $i = \gamma(j)$, or of genus 2 otherwise, such that the i^{th} and j^{th} basis elements are supported on it, as well as assume that i is odd, and that $\mathrm{H}_1(\Sigma_0, \mathbb{Z})$ is spanned by a symplectic basis $\alpha_1, \beta_1, \alpha_2, \beta_2$. Then using that ρ is a homomorphism and (5.1.6) one shows that $\rho(\tau_{\alpha_1}^{-1}) = \Sigma^{1,2}$, $\rho(\tau_{\alpha_2}^{-1}\tau_{\alpha_1}^{-1}\tau_{\alpha_1+\beta_2}) = \Sigma^{1,3}$ and $\rho(\tau_{\alpha_2}\tau_{\alpha_1}\tau_{\alpha_1+\beta_2}^{-1}) = \Sigma^{3,2}$. □

5.1.2. Construction. In this section we construct the monodromy invariant of our symplectic manifold (M, σ), c.f. Definition 5.1.9. To any group homomorphism

$$f\colon \mathrm{H}_1(\Sigma, \mathbb{Z}) \to T,$$

we can assign the $2g$-tuple $(f(\alpha_i),\ f(\beta_i))_{i=1}^g$, where α_i, β_i, $1 \le i \le g$ is a basis of the homology group $\mathrm{H}_1(\Sigma, \mathbb{Z})$ satisfying formulas (5.1.3). Conversely, given a $2g$-tuple $(a_1, b_1, \dots, a_g, b_g) \in T^{2g}$, the commutativity of T implies, by the universal property of free abelian groups [**14**, Ch. I.3] that there exists a unique group homomorphism from the homology group $\mathrm{H}_1(\Sigma, \mathbb{Z})$ into the torus T which sends α_i to a_i and β_j to b_j, for all values of i, j with

$1 \leq i,\, j \leq g$. Notice that this tuple depends on the choice of basis. For each such basis we have an isomorphism of groups

$$f \mapsto (f(\alpha_i),\, f(\beta_i))_{i=1}^g, \tag{5.1.7}$$

between the homomorphism group $\mathrm{Hom}(\mathrm{H}_1(\Sigma,\, \mathbb{Z}),\, T)$ and the Cartesian product T^{2g}.

Definition 5.1.5. *Let T be a torus and m a positive integer. Let $\mathcal{H}$ be a subgroup of $\mathrm{GL}(m,\, \mathbb{Z}) \simeq \mathrm{Aut}(\mathbb{Z}^m)$. A matrix A in $\mathcal{H}$ acts on the Cartesian product T^m by sending an m-tuple x to $x \cdot A^{-1} \in T^m$ by identifying x with a homomorphism from $\mathbb{Z}^m$ to T. We say that two m-tuples $x,\, y \in T^m$ are $\mathcal{H}$-equivalent if they lie in the same $\mathcal{H}$-orbit, i.e. if there exists a matrix $A \in \mathcal{H}$ such that $y = x \cdot A$. We write $\mathcal{H} \cdot x$ for the $\mathcal{H}$-orbit of x, and $T^m/\mathcal{H}$ for the set of all $\mathcal{H}$-orbits.*

Lemma 5.1.6. *Let T be a torus, let Σ be a compact, connected, smooth orientable surface, let α_i, β_i and α'_i, β'_i be symplectic bases of the integral homology group $\mathrm{H}_1(\Sigma,\, \mathbb{Z})$, and let $f\colon\, \mathrm{H}_1(\Sigma,\, \mathbb{Z}) \to T$ be a group homomorphism. Then there exists a matrix in the symplectic group $\mathrm{Sp}(2g,\, \mathbb{Z})$ whose action in the sense of Definition 5.1.5 takes the image tuple $(f(\alpha_i),\, f(\beta_i)) \in T^{2g}$ to the image tuple $(f(\alpha'_i),\, f(\beta'_i)) \in T^{2g}$.*

Proof. Because any two symplectic bases α_i, β_i and α'_i, β'_i of $\mathrm{H}_1(\Sigma,\, \mathbb{Z})$ are taken onto each other by an element of the symplectic group $\mathrm{Sp}(2g,\, \mathbb{Z})$, the change of basis matrix from the basis α_i, β_i, to the basis α'_i, β'_i is in the group $\mathrm{Sp}(2g,\, \mathbb{Z})$. Notice that the $(2j-1)^{\mathrm{th}}$-column of this matrix consists of the coordinates of α_j, with respect to the basis α'_i, β'_i, and its $(2j)^{\mathrm{th}}$-column consists of the coordinates of β'_j. Hence we have that the tuple $(f(\alpha_i), f(\beta_i))_{i=1}^g$ is obtained by applying such a matrix to the tuple $(f(\alpha'_i), f(\beta'_i))_{i=1}^g$. □

Lemma 5.1.6 shows that the assignment

$$f \mapsto \mathrm{Sp}(2g, \mathbb{Z}) \cdot (f(\alpha_i),\, f(\beta_i))_{i=1}^g \tag{5.1.8}$$

induced by expression (5.1.7) is well defined independently of the choice of basis α_i, β_i, $1 \leq i \leq g$ of $\mathrm{H}_1(\Sigma,\, \mathbb{Z})$, as long as it is a symplectic basis. The following is a consequence of Lemma 5.1.6 and Definition 5.1.5.

Lemma 5.1.7. *Let $(M,\, \sigma)$ be a compact connected $2n$-dimensional symplectic manifold equipped with a free symplectic action of a $(2n-2)$-dimensional torus T, for which at least one, and hence every T-orbit is a symplectic submanifold of $(M,\, \sigma)$. If α_i, β_i and α'_i, β'_i, $1 \leq i \leq g$, are symplectic bases of $\mathrm{H}_1(M/T,\, \mathbb{Z})$, then for every homomorphism $f\colon\, \mathrm{H}_1(M/T,\, \mathbb{Z}) \to T$, we have that the $\mathrm{Sp}(2g,\, \mathbb{Z})$-orbits of the tuples $(f(\alpha_i),\, f(\beta_i))_{i=1}^g$ and $(f(\alpha'_i), f(\beta'_i))_{i=1}^g$ are equal.*

Remark 5.1.8. Let T be a torus. Let g be a non-negative integer, and let Σ, Σ' be compact, connected, oriented, smooth surfaces of the same genus g.

Let α_i, β_i and α'_i, β'_i respectively be symplectic bases of the integral homology groups $\mathrm{H}_1(\Sigma,\, \mathbb{Z})$ and $\mathrm{H}_1(\Sigma',\, \mathbb{Z})$. Let $G\colon \mathrm{H}_1(\Sigma,\, \mathbb{Z}) \to \mathrm{H}_1(\Sigma',\, \mathbb{Z})$ be an isomorphism which preserves the symplectic structure. Let $f\colon \mathrm{H}_1(\Sigma,\, \mathbb{Z}) \to T$ and $f'\colon \mathrm{H}_1(\Sigma',\, \mathbb{Z}) \to T$ be group homomorphisms such that $f' = f \circ G^{-1}$. Then there exists a matrix in $\mathrm{Sp}(2g,\, \mathbb{Z})$ whose action in the sense of Definition 5.1.5 sends $(f(\alpha_i),\, f(\beta_i))_{i=1}^g \in T^{2g}$ to $(f'(\alpha'_i), f'(\beta'_i))_{i=1}^g \in T^{2g}$. Conversely, if $f\colon \mathrm{H}_1(\Sigma,\, \mathbb{Z}) \to T$ and $f'\colon \mathrm{H}_1(\Sigma',\, \mathbb{Z}) \to T$ are such that there exists a matrix in $\mathrm{Sp}(2g,\, \mathbb{Z})$ sending $(f(\alpha_i),\, f(\beta_i))_{i=1}^g \in T^{2g}$ to $(f'(\alpha'_i), f'(\beta'_i))_{i=1}^g \in T^{2g}$, then there exists an isomorphism $G\colon \mathrm{H}_1(\Sigma,\, \mathbb{Z}) \to \mathrm{H}_1(\Sigma',\, \mathbb{Z})$ which preserves the symplectic structure and such that $f' = f \circ G^{-1}$.

Indeed, let A be the matrix of G in the bases α_i, β_i and α'_i, β'_i. Then $A \in \mathrm{Sp}(2g,\, \mathbb{Z})$ and $(f(\alpha_i),\, f(\beta_i))_{i=1}^{2g} = (f(\alpha'_i),\, f(\beta'_i))_{i=1}^{2g} \cdot A^{-1}$. There is a commutative diagram:

$$\begin{tikzcd} \mathbb{Z}^{2g} \arrow[r, "(\alpha_i,\beta_i)"] \arrow[d, "A"] & \mathrm{H}_1(\Sigma,\, \mathbb{Z}) \arrow[r, "f"] \arrow[d, "G"] & T \\ \mathbb{Z}^{2g} \arrow[r, "(\alpha'_i,\beta'_i)"] & \mathrm{H}_1(\Sigma',\, \mathbb{Z}) \arrow[ur, "f'"] & \end{tikzcd} \tag{5.1.9}$$

where the composite of the two top maps equals the mapping $(f(\alpha_i),\, f(\beta_i))$ and the composite of the two bottom maps is $(f'(\alpha'_i),\, f'(\beta'_i))$.

DEFINITION 5.1.9. *Let $(M,\, \sigma)$ be a compact connected $2n$-dimensional symplectic manifold equipped with a free symplectic action of a torus T for which at least one, and hence every T-orbit is a symplectic submanifold of $(M,\, \sigma)$. The* free monodromy invariant of $(M,\, \sigma,\, T)$ *is the* $\mathrm{Sp}(2g,\, \mathbb{Z})$*-orbit*

$$\mathrm{Sp}(2g,\, \mathbb{Z}) \cdot (\mu_{\mathrm{h}}(\alpha_i),\, \mu_{\mathrm{h}}(\beta_i))_{i=1}^g \in T^{2g}/\mathrm{Sp}(2g,\, \mathbb{Z}),$$

where α_i, β_i, $1 \le i \le g$, is a basis of the homology group $\mathrm{H}_1(M/T,\, \mathbb{Z})$ satisfying expression (5.1.3) and μ_{h} is the homomorphism induced on homology by the monodromy homomorphism μ of the connection of symplectic orthogonal complements to the tangent spaces to the T-orbits (c.f. Proposition 2.4.1) by means of the Hurewicz map (c.f. formula (4.3.1) and Definition 5.1.5).

5.2. Uniqueness

5.2.1. List of ingredients of $(M,\, \sigma,\, T)$. We start with the following definition.

DEFINITION 5.2.1. *Let $(M,\, \sigma)$ be a compact connected $2n$-dimensional symplectic manifold equipped with a free symplectic action of a $(2n-2)$--dimensional torus T for which at least one, and hence every T-orbit is a symplectic submanifold of $(M,\, \sigma)$. The* list of ingredients of $(M,\, \sigma,\, T)$ *consists of the following items.*

1) *The genus g of the surface M/T (c.f. Remark 2.3.6).*

2) *The total symplectic area of the symplectic surface* $(M/T, \nu)$*, where the symplectic form* ν *is defined by the condition* $\pi^*\nu|_{\Omega_x} = \sigma|_{\Omega_x}$ *for every* $x \in M$*,* $\pi \colon M \to M/T$ *is the projection map, and where for each* $x \in M$*,* $\Omega_x = (\mathrm{T}_x(T \cdot x))^{\sigma_x}$ *(c.f. Lemma 3.2.1).*
3) *The unique non-degenerate antisymmetric bilinear form* $\sigma^{\mathfrak{t}} \colon \mathfrak{t} \times \mathfrak{t} \to \mathbb{R}$ *on the Lie algebra* $\mathfrak{t}$ *of* T *such that for all* $X, Y \in \mathfrak{t}$ *and all* $x \in M$ $\sigma_x(X_M(x), Y_M(x)) = \sigma^{\mathfrak{t}}(X, Y)$ *(c.f. Lemma 2.1.1).*
4) *The free monodromy invariant of* (M, σ, T)*, i.e. the* $\mathrm{Sp}(2g, \mathbb{Z})$*-orbit*
$$\mathrm{Sp}(2g, \mathbb{Z}) \cdot (\mu_{\mathrm{h}}(\alpha_i), \mu_{\mathrm{h}}(\beta_i))_{i=1}^{g}$$
of the $2g$*-tuple* $(\mu_{\mathrm{h}}(\alpha_i), \mu_{\mathrm{h}}(\beta_i))_{i=1}^{g} \in T^{2g}$*, c.f. Definition 5.1.9.*

5.2.2. Uniqueness Statement. The next two results say that the list of ingredients of (M, σ, T) as in Definition 5.2.1 is a complete set of invariants of (M, σ, T). We start with a preliminary remark.

REMARK 5.2.2. Let T be a torus. Let $p \colon X \to B$, $p' \colon X' \to B'$ be smooth principal T-bundles equipped with flat connections Ω, Ω', and let $\Phi \colon X \to X'$ be a T-bundle isomorphism such that $\Phi^*\Omega' = \Omega$ (i.e. $\Omega'_{\Phi(x)} = \mathrm{T}_x\,\Phi(\Omega_x)$ for each $x \in X$). Let $x_0 \in X$. Let $x'_0 := \Phi(x_0)$, and let $\widehat{\Phi} \colon B \to B'$ be induced by Φ and such that $\widehat{\Phi}(b_0) = b'_0$, where $p(x_0) = b_0$, $p'(x'_0) = b'_0$. The monodromy homomorphism $\mu \colon \pi_1(B, b_0) \to T$ associated to the connection Ω is the unique homomorphism such that

$$\lambda_\gamma(1) = \mu([\gamma]) \cdot x_0, \tag{5.2.1}$$

for every path $\gamma \colon [0, 1] \to B$ such that $\gamma(0) = \gamma(1) = b_0$, where λ_γ is the unique horizontal lift of γ with respect to the connection Ω, such that $\lambda_\gamma(0) = x_0$.

Applying Φ to both sides of expression (5.2.1), and using the fact that Φ preserves the horizontal subspaces, denoting by $\lambda'_{\gamma'}$ the unique horizontal lift with respect to Ω', of any loop $\gamma' \colon [0, 1] \to B'$ with $\gamma'(0) = \gamma'(1) = b'_0$, we obtain that $\lambda'_{\widehat{\Phi}\,\gamma}(1) = (\mu \circ (\widehat{\Phi}_*)^{-1})([\widehat{\Phi}\,\gamma]) \cdot x'_0$. Here $\widehat{\Phi}_* \colon \pi_1(B, b_0) \to \pi_1(B', b'_0)$ is the isomorphism induced by $\widehat{\Phi}$. Hence by uniqueness of the monodromy homomorphism we have that $\mu' = \mu \circ (\widehat{\Phi}_*)^{-1} \colon \pi_1(B', b_0) \to T$, and since $\mu = \mu_{\mathrm{h}} \circ h_1$ and $\mu' = \mu_{\mathrm{h}} \circ h_1$, see expression (4.3.1), that $\mu'_{\mathrm{h}} = \mu_{\mathrm{h}} \circ (\widehat{\Phi}_*)^{-1} \colon \pi_1(B', b_0) \to T$, where in this case $\widehat{\Phi}_* \colon \mathrm{H}_1(B, \mathbb{Z}) \to \mathrm{H}_1(B', \mathbb{Z})$ is the homomorphism induced by $\widehat{\Phi}$ in homology.

LEMMA 5.2.3. *Let* (M, σ) *be a compact connected* $2n$*-dimensional symplectic manifold equipped with a free symplectic action of a* $(2n-2)$*-dimensional torus* T *for which at least one, and hence every* T*-orbit is a symplectic submanifold of* (M, σ)*. If* (M', σ') *is a compact connected* $2n$*-dimensional symplectic manifold equipped with a free symplectic action of* T *for which at least one, and hence every* T*-orbit is a* $(2n-2)$*-dimensional symplectic submanifold of* (M', σ')*, and* (M', σ') *is* T*-equivariantly symplectomorphic* *to* (M, σ)*, then the list of ingredients of* (M', σ', T) *is equal to the list of ingredients of* (M, σ, T)*.*

Proof. Let Φ be a T-equivariant symplectomorphism from (M, σ) to (M', σ'). Like in the proof of Lemma 3.2.2, the mapping Φ descends to a symplectomorphism $\widehat{\Phi}$ from the orbit space $(M/T, \nu)$ onto the orbit space $(M'/T, \nu')$. By Remark 2.3.6, the orbit spaces M/T and M'/T are compact, connected, orientable, smooth surfaces, and because they are diffeomorphic, M/T and M'/T must have the same genus g.

Lemma 3.2.2 and Remark 3.2.3 imply that ingredient 2) of (M, σ) equals ingredient 2) of (M', σ').

If $X, Y \in \mathfrak{t}$, then the T-equivariance of Φ implies that $\Phi^*(X_{M'}) = X_M$, $\Phi^*(Y_{M'}) = Y_M$. In combination with $\sigma = \widehat{\Phi}^*\sigma'$, this implies, in view of Lemma 2.1.1, that

$$\sigma^{\mathfrak{t}}(X, Y) = \Phi^*(\sigma'(X_{M'}, Y_{M'})) = \Phi^*((\sigma')^{\mathfrak{t}}(X, Y)) = (\sigma')^{\mathfrak{t}}(X, Y),$$

where we have used in the last equation that $(\sigma')^{\mathfrak{t}}(X, Y)$ is a constant on M'. This proves that $\sigma^{\mathfrak{t}} = (\sigma')^{\mathfrak{t}}$. Since $\Phi^*\Omega' = \Omega$, we have that

$$\mu'_{\mathrm{h}} = \mu_{\mathrm{h}} \circ (\widehat{\Phi}_*)^{-1},$$

as mappings from the orbifold homology group $\mathrm{H}_1(M'/T, \mathbb{Z})$ into the torus T, where the mapping $\widehat{\Phi}_*$ from the group $\mathrm{H}_1(M/T, \mathbb{Z})$ to $\mathrm{H}_1(M'/T, \mathbb{Z})$ is the group isomorphism induced by the orbifold diffeomorphism $\widehat{\Phi}$ from the orbit space M/T onto M'/T (see Remark 5.2.2). By Lemma 5.1.4, $\widehat{\Phi}_*$ preserves the intersection form, and therefore the images under $\widehat{\Phi}_*$ of α_i, β_i, $1 \leq i \leq g$, which we will call α'_i, β'_i, form a symplectic basis of $\mathrm{H}_1(M'/T, \mathbb{Z})$. Hence $\mu_{\mathrm{h}}(\alpha_i) = \mu'_{\mathrm{h}}(\alpha'_i)$ and $\mu_{\mathrm{h}}(\beta_i) = \mu'_{\mathrm{h}}(\beta'_i)$, which in turn implies that ingredient 4) of (M, σ) equals ingredient 4) of (M', σ'). □

Proposition 5.2.4. *Let* (M, σ) *be a compact connected* $2n$*-dimensional symplectic manifold equipped with a free symplectic action of a* $(2n-2)$*--dimensional torus* T *for which at least one, and hence every* T*-orbit is a symplectic submanifold of* (M, σ)*. Then if* (M', σ') *is a compact connected* $2n$*-dimensional symplectic manifold equipped with a free symplectic action of* T *for which at least one, and hence every* T*-orbit is a* $(2n-2)$*-dimensional symplectic submanifold of* (M', σ')*, and the list of ingredients of* (M, σ, T) *is equal to the list of ingredients of* (M', σ', T)*, then* (M, σ) *is* T*-equivariantly symplectomorphic to* (M', σ')*.*

Proof. Suppose that the list of ingredients of (M, σ) equals the list of ingredients of (M', σ'). Let α_i, β_i, and α'_i, β'_i be, respectively, symplectic bases of the first integral homology groups $\mathrm{H}_1(M/T, \mathbb{Z})$ and $\mathrm{H}_1(M'/T, \mathbb{Z})$, and suppose that ingredient 4) of (M, σ) is equal to ingredient 4) of (M', σ'). Let μ, μ', μ_{h}, μ'_{h} be the corresponding homomorphisms respectively associated to (M, σ), (M', σ') as in Definition 4.3.1. Then, by Definition 5.1.5, there exists a matrix in the integer symplectic group $\mathrm{Sp}(2g, \mathbb{Z})$ which takes the tuple of images of the α_i, β_i under μ_{h} to the tuple of images of α_i', β_i' under μ_{h}'. This means that

$$\mu_{\mathrm{h}} = \mu_{\mathrm{h}}' \circ G, \tag{5.2.2}$$

as maps from the homology group $\mathrm{H}_1(M/T, \mathbb{Z})$ to the torus T, where the mapping G is the intersection form preserving automorphism from the homology group $\mathrm{H}_1(M/T, \mathbb{Z})$ to $\mathrm{H}_1(M'/T, \mathbb{Z})$ whose matrix with respect to the bases α_i, β_i and α_i', β_i' is precisely the aforementioned matrix. Since M/T and M'/T have the same genus, by Lemma 5.1.4, there exists a surface diffeomorphism $F\colon M/T \to M'/T$ such that $F_* = G$, and hence by (5.2.2),

$$\mu = \mu' \circ F_*. \tag{5.2.3}$$

Let ν and ν' be the symplectic forms given by Lemma 3.2.1. If $\nu_0 := F^*\nu'$, the symplectic manifold $(M/T,\, \nu_0)$ is symplectomorphic to $(M'/T,\, \nu')$, by means of F. Let $\widetilde{\nu_0}$ be the pullback of the 2-form ν_0 by the universal covering map $\psi\colon \widetilde{M/T} \to M/T$ of M/T based at p_0, and similarly we define $\widetilde{\nu'}$ by means of the universal cover $\psi'\colon \widetilde{M'/T} \to M'/T$ based at $F(p_0)$. Choose x_0, x_0' such that $p_0 = \psi(x_0)$, $p_0' = \psi'(x_0')$. By standard covering space theory, the symplectomorphism F between $(M/T,\, \nu_0)$ and $(M'/T,\, \nu')$ lifts to a unique symplectomorphism $\widetilde{F}$ between $(\widetilde{M/T},\, \widetilde{\nu_0})$ and $(\widetilde{M'/T},\, \widetilde{\nu'})$ such that $\widetilde{F}(x_0) = x_0'$. Now, let σ^T be the unique translation invariant symplectic form on the torus T which is uniquely determined by the antisymetric bilinear form $\sigma^{\mathfrak{t}}$, which since ingredient 3) of $(M,\, \sigma)$ is equal to ingredient 3) of $(M',\, \sigma')$, equals the antisymetric bilinear form $(\sigma')^{\mathfrak{t}}$. Then the product $\widetilde{M/T} \times T$ is T-equivariantly symplectomorphic to the product $\widetilde{M'/T} \times T$ by means of the map

$$([\gamma],\, t) \mapsto (\widetilde{F}([\gamma]),\, t), \tag{5.2.4}$$

where T is acting by translations on the right factor of both spaces, $\widetilde{M/T} \times T$ is equipped with the symplectic form $\widetilde{\nu_0} \oplus \sigma^T$, and $\widetilde{M'/T} \times T$ is equipped with the symplectic form $\widetilde{\nu'} \oplus \sigma^T$. As in Definition 3.3.1, let $[\delta] \in \pi_1(M/T,\, p_0)$ act diagonally on the product $\widetilde{M/T} \times T$ by sending the pair $([\gamma],\, t)$ to $([\gamma\, \delta^{-1}],\, \mu([\delta])\, t)$, and similarly let $[\delta'] \in \pi_1(M'/T,\, p_0)$ act on $\widetilde{M'/T} \times T$ by sending $([\gamma'],\, t)$ to $([\gamma'\, (\delta')^{-1}], \mu'([\delta']) \cdot t)$, hence giving rise to the quotient spaces $\widetilde{M/T} \times_{\pi_1(M/T,\, p_0)} T$ and $\widetilde{M'/T} \times_{\pi_1(M'/T,\, p_0')} T$. (Here μ and μ' are the monodromy homomorphisms respectively associated to the connections Ω and Ω'). Because the based fundamental groups $\pi_1(M/T,\, p_0)$ and $\pi_1(M'/T,\, p_0')$ act properly and discontinuously, both the action of T on the products, as well as the symplectic forms, induce well-defined actions and symplectic forms on these quotients. Therefore, because of expression

(5.2.3), the assignment induced by the product mapping (5.2.4)

$$\Psi\colon [[\gamma],\, t]_{\pi_1(M/T,\,p_0)} \mapsto [\widetilde{F}([\gamma]),\, t]_{\pi_1(M'/T,\,p_0')},$$

is a T-equivariant symplectomorphism between $\widetilde{M/T} \times_{\pi_1(M/T,\,p_0)} T$ and $\widetilde{M'/T} \times_{\pi_1(M'/T,\,p_0')} T$. Because M/T and M'/T have the same genus and symplectic area, by Moser's theorem, c.f. [**41**, Th. 3.17], the (compact, connected, smooth, orientable) surfaces $(M/T,\, \nu)$ and $(M'/T,\, \nu')$ are symplectomorphic, and hence $(M/T,\, \nu)$ is symplectomorphic to $(M/T,\, \nu_0)$. By Theorem 3.4.3, $(M,\, \sigma)$ is T-equivariantly symplectomorphic to the associated bundle $\widetilde{M/T} \times_{\pi_1(M/T,\,p_0)} T$, with the symplectic form $\widetilde{\widetilde{\nu} \oplus \sigma^T}$, by means of a T-equivariant symplectomorphism ϕ, and hence T-equivariantly symplectomorphic to the associated bundle $\widetilde{M/T} \times_{\pi_1(M/T,\,p_0)} T$ with the symplectic form $\widetilde{\widetilde{\nu_0} \oplus \sigma^T}$, say by means of $\widetilde{\phi}$. Similarly, $(M',\, \sigma')$ is T-equivariantly symplectomorphic to $\widetilde{M'/T} \times_{\pi_1(M'/T,\,p_0')} T$, by means of a T-equivariant symplectomorphism φ, and therefore the composite map $\varphi \circ \Psi \circ \widetilde{\phi}^{-1}\colon M \to M'$ is a T-equivariant symplectomorphism between $(M,\, \sigma)$ and $(M',\, \sigma')$. $\square$

5.3. Existence

5.3.1. List of ingredients for T. We start by making an abstract list of ingredients which we associate to a torus T.

Definition 5.3.1. *Let T be a torus. The* list of ingredients for T *consists of the following items.*

i) *A non-negative integer g.*
ii) *A positive real number λ.*
iii) *An non-degenerate antisymmetric bilinear form $\sigma^{\mathfrak{t}}$ on the Lie algebra $\mathfrak{t}$ of T.*
iv) *A $\mathrm{Sp}(2g,\, \mathbb{Z})$-orbit $\gamma \in T^{2g}/\mathrm{Sp}(2g,\, \mathbb{Z})$, where $\mathrm{Sp}(2g,\, \mathbb{Z})$ denotes the group of $2g$-dimensional square symplectic matrices with integer entries, c.f. Definition 5.1.5.*

5.3.2. Existence Statement. Any list of ingredients as in Definition 5.3.1 gives rise to one of our manifolds with symplectic T-action.

Proposition 5.3.2. *Let T be a $(2n-2)$-dimensional torus. Then given a list of ingredients for T, as in Definition 5.3.1, there exists a compact connected $2n$-dimensional symplectic manifold $(M,\, \sigma)$ with a free symplectic action of T for which at least one, and hence every T-orbit is a symplectic submanifold of $(M,\, \sigma)$, and such that the list of ingredients of $(M,\, \sigma,\, T)$ in Definition 5.2.1 is equal to the list of ingredients for T in Definition 5.3.1.*

Proof. Let $\mathcal{I}$ be a list of ingredients for the torus T, as in Definition 5.3.1. Let the pair $(\Sigma,\, \sigma^{\Sigma})$ be a compact, connected symplectic surface of genus g given by ingredient i) of $\mathcal{I}$ in Definition 5.3.1, and with total symplectic area equal to the positive real number λ given given by ingredient ii) of $\mathcal{I}$. Let the space $\widetilde{\Sigma}$ be the universal cover of Σ based at an arbitrary regular point $p_0 \in \Sigma$, which we fix for the rest of the proof. Let T be the $(2n-2)$-dimensional torus that we started with, equipped with the unique T-invariant symplectic form σ^T on T whose associated non-degenerate antisymmetric bilinear form is $\sigma^{\mathfrak{t}}\colon \mathfrak{t} \times \mathfrak{t} \to \mathbb{R}$, given by ingredient iii) of $\mathcal{I}$. Write ingredient iv) of Definition 5.3.1 as $\gamma = \mathrm{Sp}(2g,\, \mathbb{Z}) \cdot (a_i,\, b_i)_{i=1}^{g} \in T^{2g}/\mathrm{Sp}(2g,\, \mathbb{Z})$, and recall the existence [**21**, Ex. 2A.1] of a symplectic $\mathbb{Z}$-basis $\alpha_i,\, \beta_i,\, 1 \leq i \leq g$, of the integral homology group $\mathrm{H}_1(\Sigma,\, \mathbb{Z})$, satisfying expression (5.1.3). Let μ_{h} be the unique homomorphism from $\mathrm{H}_1(\Sigma,\, \mathbb{Z})$ into T such that

$$\mu_{\mathrm{h}}(\alpha_i) := a_i, \quad \mu_{\mathrm{h}}(\beta_i) := b_i,$$

for all $1 \leq i \leq 2g$. Define $\mu := \mu_{\mathrm{h}} \circ h_1$, where h_1 denotes the Hurewicz homomorphism from $\pi_1(\Sigma,\, p_0)$ to $\mathrm{H}_1(\Sigma,\, \mathbb{Z})$. μ is a homomorphism from $\pi_1(\Sigma,\, p_0)$ into T. Let the fundamental group $\pi_1(\Sigma,\, p_0)$ act on the Cartesian product $\widetilde{\Sigma} \times T$ by the diagonal action $[\delta]\,([\gamma],\, t) = ([\delta\, \gamma^{-1}],\, \mu([\delta]) \cdot t)$. We equip the universal cover $\widetilde{\Sigma}$ with the symplectic form $\sigma^{\widetilde{\Sigma}}$ obtained as the pullback of σ^{Σ} by the universal covering mapping $\widetilde{\Sigma} \to \Sigma$, and we equip the product space $\widetilde{\Sigma} \times T$ with the product symplectic form $\sigma^{\widetilde{\Sigma}} \oplus \sigma^T$, and let the torus T act by translations on the right factor of $\widetilde{\Sigma} \times T$. Define the associated bundle[1]

$$M^{\Sigma}_{\mathrm{model}} := \widetilde{\Sigma} \times_{\pi_1(\Sigma,\, p_0)} T. \tag{5.3.1}$$

Because $\pi_1(\Sigma,\, p_0)$ is acting properly and discontinuously, the symplectic form on $\widetilde{\Sigma} \times T$ passes to a unique symplectic form $\sigma^{\Sigma}_{\mathrm{model}} := \widetilde{\sigma^{\widetilde{\Sigma}} \oplus \sigma^T}$ on $M^{\Sigma}_{\mathrm{model}}$ (as in the proof of Theorem 3.4.3). Similarly, the action of the torus T by translations on the right factor of $\widetilde{\Sigma} \times T$ passes to an action of T on $M^{\Sigma}_{\mathrm{model}}$, which is free. It follows from the construction that $(M^{\Sigma}_{\mathrm{model}},\, \sigma^{\Sigma}_{\mathrm{model}})$ is a compact, connected symplectic manifold, with a free T-action for which every T-orbit is a $\dim T$-dimensional symplectic submanifold of $(M^{\Sigma}_{\mathrm{model}},\, \sigma^{\Sigma}_{\mathrm{model}})$.

We have left to show that the list $\mathcal{I}$ equals the list of ingredients of $(M^{\Sigma}_{\mathrm{model}},\, \sigma^{\Sigma}_{\mathrm{model}})$. Since the action of the torus T on $M^{\Sigma}_{\mathrm{model}}$ is induced by the action of T on the right factor of $\widetilde{\Sigma} \times T$, M_{model}/T is symplectomorphic to $\widetilde{\Sigma}/\pi_1(\Sigma,\, p_0)$ with the symplectic form induced by the mapping $\widetilde{\Sigma} \to \widetilde{\Sigma}/\pi_1(\Sigma,\, p_0)$, which by construction (i.e. by regularity of the orbifold universal cover) is symplectomorphic to $(\Sigma,\, \sigma^{\Sigma})$. Therefore $M^{\Sigma}_{\mathrm{model}}/T$ with the symplectic form $\nu^{\Sigma}_{\mathrm{model}}$ given by Lemma 3.2.1, is symplectomorphic to $(\Sigma,\, \sigma^{\Sigma})$, and in particular the total symplectic area of $(M^{\Sigma}_{\mathrm{model}}/T,\, \nu^{\Sigma}_{\mathrm{model}})$

[1] We should probably write $M^{\Sigma}_{\mathrm{model},\, p_0}$ instead of $M^{\Sigma}_{\mathrm{model}}$, but we avoid to write the dependance on p_0 to shorten the notation, and since the models are identified for all choices of p_0.

equals the total symplectic area of $(\Sigma, \sigma^{\Sigma})$, which is equal to the positive real number λ. Let $p\colon \widetilde{\Sigma} \times T \to M_{\mathrm{model}}$ be the projection map. It follows from the definition of $\sigma^{\Sigma}_{\mathrm{model}}$ that for all $X, Y \in \mathfrak{t}$, the real number

$$(\sigma^{\Sigma}_{\mathrm{model}})_{[[\gamma],\, t]_{\pi_1(\Sigma,\, p_0)}}(X_{M^{\Sigma}_{\mathrm{model}}}([[\gamma],\, t]_{\pi_1(\Sigma,\, p_0)}), Y_{M^{\Sigma}_{\mathrm{model}}}([[\gamma],\, t]_{\pi_1(\Sigma,\, p_0)}))$$

is equal to

$$(\widetilde{\sigma^{\widetilde{\Sigma}} \oplus \sigma^{T}})_{[[\gamma],\, t]_{\pi_1(\Sigma,\, p_0)}}(\mathrm{T}_{([\gamma],\, t)}\, p(X_{\widetilde{\Sigma}\times T}([\gamma],\, t)),\ \mathrm{T}_{([\gamma],\, t)}\, p(Y_{\widetilde{\Sigma}\times T}([\gamma],\, t)))$$

which is equal to

$$\begin{aligned}(\sigma^{\widetilde{\Sigma}} \oplus \sigma^{T})_{([\gamma],\, t)}(X_{\widetilde{\Sigma}\times T}([\gamma],\, t)),\, Y_{\widetilde{\Sigma}\times T}([\gamma],\, t)) &= \sigma^{T}_{t}(X_T(t),\, Y_T(t)) \\ &= \sigma^{\mathfrak{t}}(X,\, Y). \end{aligned} \tag{5.3.2}$$

In the first equality of (5.3.2) we have used that the vector fields $X_{\widetilde{\Sigma}\times T}$, $Y_{\widetilde{\Sigma}\times T}$ are tangent to the T-orbits $\{u\} \times T$ of $\widetilde{\Sigma} \times T$, and the symplectic form vanishes on the orthogonal complements $\mathrm{T}_{(u,\, t)}(\widetilde{\Sigma} \times \{t\})$ to the T-orbits. The last equality follows from the definition of $\sigma^{\mathfrak{t}}$.

Finally let $\Omega^{\Sigma}_{\mathrm{model}}$ stand for the flat connection on $M^{\Sigma}_{\mathrm{model}}$ given by the symplectic orthogonal complements to the tangent spaces to the orbits of the T-action, see Proposition 2.4.1, and let $\mu_{\mathrm{h}}^{\Omega^{\Sigma}_{\mathrm{model}}}$ stand for the induced homomorphism $\mu_{\mathrm{h}}^{\Omega^{\Sigma}_{\mathrm{model}}}\colon \mathrm{H}_1(M^{\Sigma}_{\mathrm{model}}/T,\, \mathbb{Z}) \to T$ in homology. If $f\colon \mathrm{H}_1(M^{\Sigma}_{\mathrm{model}}/T,\, \mathbb{Z}) \to \mathrm{H}_1(\Sigma,\, \mathbb{Z})$ is the group isomorphism induced by the symplectomorphism

$$M^{\Sigma}_{\mathrm{model}}/T \to \widetilde{\Sigma}/\pi_1(\Sigma,\, p_0) \to \Sigma, \tag{5.3.3}$$

where each arrow in (5.3.3) represents the natural map, we have that

$$\mu_{\mathrm{h}}^{\Omega^{\Sigma}_{\mathrm{model}}} = \mu_{\mathrm{h}} \circ f. \tag{5.3.4}$$

Because f is induced by a diffeomorphism, by Lemma 5.1.4 f preserves the intersection form and hence the unique collection of elements α'_i, β'_i, $1 \le i \le g$ such that $f(\alpha'_i) = \alpha_i$ and $f(\beta'_i) = \beta_i$, for all $1 \le i \le g$, is a symplectic basis of the homology group $\mathrm{H}_1(M^{\Sigma}_{\mathrm{model}}/T,\, \mathbb{Z})$. Let $\widehat{\gamma}$ be the $2g$-tuple of elements $\mu_{\mathrm{h}}^{\Omega^{\Sigma}_{\mathrm{model}}}(\alpha'_i)$, $\mu_{\mathrm{h}}^{\Omega^{\Sigma}_{\mathrm{model}}}(\beta'_i)$, $1 \le i \le g$. Therefore by (5.3.4)

$$\widehat{\gamma} = (\mu_{\mathrm{h}}(\alpha_i),\, \mu_{\mathrm{h}}(\beta_i))_{i=1}^{g}.$$

The result follows because the $\mathrm{Sp}(2g,\, \mathbb{Z})$-orbit of $(\mu_{\mathrm{h}}(\alpha_i),\, \mu_{\mathrm{h}}(\beta_i))_{i=1}^{g}$ is equal to item 4) in Definition 5.2.1. □

5.4. Classification theorem

We state and prove the two main results of Chapter 5, by putting together previous results.

THEOREM 5.4.1. *Let T be a $(2n-2)$-dimensional torus. Let (M, σ) be a compact connected $2n$-dimensional symplectic manifold on which T acts freely and symplectically and such that at least one, and hence every T-orbit is a symplectic submanifold of (M, σ).*

Then the list of ingredients of (M, σ, T) as in Definition 5.2.1 is a complete set of invariants of (M, σ, T), in the sense that, if (M', σ') is a compact connected $2n$-dimensional symplectic manifold equipped with a free symplectic action of T for which at least one, and hence every T-orbit is a symplectic submanifold of (M', σ'), (M', σ') is T-equivariantly symplectomorphic to (M, σ) if and only if the list of ingredients of (M', σ', T) is equal to the list of ingredients of (M, σ, T).

And given a list of ingredients for T, as in Definition 5.3.1, there exists a symplectic $2n$-dimensional manifold (M, σ) with a free symplectic action of T for which at least one, and hence every T-orbit is a symplectic submanifold of (M, σ), such that the list of ingredients of (M, σ, T) is equal to the list of ingredients for T.

PROOF. It follows by putting together Lemma 5.2.3, Proposition 5.2.4 and Proposition 5.3.2. Observe that the combination of Lemma 5.2.3, Proposition 5.2.4 gives the uniqueness part of the theorem, while Proposition 5.3.2 gives the existence part. □

COROLLARY 5.4.2. *Let (M, σ) be a compact connected $2n$-dimensional symplectic manifold equipped with a free symplectic action of a $(2n-2)$-dimensional torus T for which at least one, and hence every T-orbit is a symplectic submanifold of (M, σ).*

Then the genus g of the surface M/T is a complete invariant of the T-equivariant diffeomorphism type of M, in the following sense. If (M', σ') is a compact connected $2n$-dimensional symplectic manifold equipped with a free symplectic action of a $(2n-2)$-dimensional torus such that at least one, and hence every T-orbit is a symplectic submanifold of (M', σ'), M' is T-equivariantly diffeomorphic to M if and only if the genus of M'/T is equal to the genus of M/T.

Moreover, given any non-negative integer, there exists a $2n$-dimensional symplectic manifold (M, σ) with a free T-action such that at least one, and hence every T-orbit is a symplectic submanifold of (M, σ), and such that the genus of M/T is precisely the aforementioned integer.

The proof of Corollary 5.4.2 is immediate. Indeed, by Corollary 4.1.2, M is T-equivariantly diffeomorphic to the Cartesian product $M/T \times T$ equipped with the action of T by translations on the right factor of the product. Suppose that Φ is a T-equivariant symplectomorphism from (M, σ) to (M', σ'). Because Φ is T-equivariant, it descends to a diffeomorphism $\widehat{\Phi}$ from the orbit space M/T onto the orbit space M'/T, and hence the genus of M/T equals the genus of M'/T. Conversely, suppose that (M, σ) and (M', σ') are such that the genus of M/T equals the genus of M'/T. By the classification theorem of compact, connected, orientable (boundaryless of course) surfaces, there exists a diffeomorphism $F\colon M/T \to M'/T$. Hence the map $M/T \times T \to M'/T \times T$ given by $(x, t) \to (F(x), t)$ is a T-equivariant diffeomorphism, and by Corollary 4.1.2 we are done. Now let g be a non-negative integer, let Σ be a (compact, connected, orientable) surface of genus g, and let T be a $(2n-2)$-dimensional torus. Then $M_g := \Sigma \times T$ is a $2n$-dimensional manifold. Equip it with any product symplectic form, and with the action of T by translations on the right factor. Then the T-orbits, which are of the form $\{u\} \times T$, $u \in \Sigma$, are symplectic submanifolds of M_g, and M_g/T is diffeomorphic to Σ by means of $[u, t] \mapsto u$.

Remark 5.4.3. We summarize Corollary 5.4.2 in the language of categories, c.f. MacLane's book [**35**]. Let T be a torus and let $\mathcal{M}$ denote the category of which the objects are the compact connected symplectic manifolds (M, σ) together with a free symplectic T-action on (M, σ), such that at least one, and hence every T-orbit is a symplectic submanifold of M, and of which the morphisms are the T-equivariant symplectomorphisms of (M, σ). Let $\mathcal{Z}^+$ denote category which consists of the set of non-negative integers, and of which the identity is the only endomorphism of categories. Then the assignment $\iota\colon M \mapsto g$, where g is the genus of the surface M/T, is a full functor of categories from the category $\mathcal{M}$ onto the category $\mathcal{Z}^+$. In particular the proper class $\mathcal{M}/\sim$ of isomorphism classes in $\mathcal{M}$ is a set, and the functor $M \mapsto g$ induces a bijective mapping $\iota/\sim$ from the category $\mathcal{M}/\sim$ onto the category $\mathcal{Z}^+$.

Let $\mathcal{I}$ denote the set of all lists of ingredients as in Definition 5.3.1, viewed as a category, and of which the identities are the only endomorphisms of categories. Then the assignment ι in Definition 5.2.1 is a full functor of categories from the category $\mathcal{M}$ onto the category $\mathcal{I}$. In particular the proper class $\mathcal{M}/\sim$ of isomorphism classes in $\mathcal{M}$ is a set, and the functor $\iota\colon \mathcal{M} \to \mathcal{I}$ induces a bijective mapping $\iota/\sim$ from $\mathcal{M}/\sim$ onto $\mathcal{I}$. The fact that the mapping $\iota\colon \mathcal{M} \to \mathcal{I}$ is a functor and the mapping $\iota/\sim$ is injective follows from the uniqueness part of Theorem 5.4.1, while the surjectivity of ι, follows from the existence part.

CHAPTER 6

Orbifold homology and geometric mappings

A tool needed to extend the results of Chapter 5 to non-free actions is Theorem 6.4.2, which is a characterization of geometric isomorphisms of orbifold homology, c.f. Definition 6.2.1. Such a classification appears to be of independent interest as it generalizes a classical result about smooth surfaces to smooth orbisurfaces.

6.1. Geometric torsion in homology of orbifolds

Compact, connected, orbisurfaces (2-dimensional orbifolds) are classified by the genus of the underlying surface and the order of the singularities, see Theorem 9.5.2 in the appendix.

Definition 6.1.1. *Let Σ be a smooth, compact, connected, orientable smooth orbisurface with n singular points. Fix an order in the singular points, say $p_1, \dots, p_n$, such that the order of p_k is less than or equal to the order of p_{k+1}. We say that a collection of n elements $\{\gamma_k\}_{k=1}^n \subset \mathrm{H}_1^{\mathrm{orb}}(\Sigma, \mathbb{Z})$ is a* geometric torsion basis[1] of $\mathrm{H}_1^{\mathrm{orb}}(\Sigma, \mathbb{Z})$ *if γ_k is the homology class of a loop $\widetilde{\gamma}_k$ obtained as an oriented boundary of a closed small disk containing the k^{th} singular point of the orbisurface Σ with respect to the ordering, and where no two such disks intersect.*

Definition 6.1.2. *Let $\vec{o} = (o_k)_{k=1}^n$ be an n-tuple of positive integers. We call $\mathrm{S}_n^{\vec{o}}$ the subgroup of the permutation group S_n which preserves the n-tuple $\vec{o}$, in a formula $\mathrm{S}_n^{\vec{o}} := \{\tau \in \mathrm{S}_n \mid (o_{\tau(k)})_{k=1}^n = \vec{o}\}$.*

Lemma 6.1.3. *Let Σ be a compact, connected, orientable, boundaryless smooth orbisurface. Assume moreover that Σ is a good orbisurface. Then the order of any cone point of Σ is equal to the order of the homotopy class of a small loop around that point.*

Proof. Take a cone point with cone angle $2\pi/n$, and let γ denote the associated natural generator of $\pi_1^{\mathrm{orb}}(\Sigma, x_0)$. We already know that $\gamma^n = 1$, because we have that relation in the presentation of the orbifold fundamental group. Since the orbifold has a manifold cover, the projection around the pre-image of the cone point is a n-fold branched cover, which implies that for any $k < n$, γ^k does not lift to the cover and so it must be non-trivial. □

[1] We use the word "torsion" because a geometric torsion basis will generate the torsion subgroup of the orbifold homology group. Similarly, we use "geometric" because the homology classes come from geometric elements, loops around singular points.

Let g, n, o_k, $1 \leq k \leq n$, be non-negative integers, and let Σ be a compact, connected, orientable smooth orbisurface with underlying topological space a surface of genus g, and with n singular points p_k of respective orders o_k. The orbifold fundamental group $\pi_1^{\mathrm{orb}}(\Sigma,\, p_0)$ has group presentation

$$(6.1.1)\langle \{\alpha_i,\, \beta_i\}_{i=1}^{g},\, \{\gamma_k\}_{k=1}^{n} \mid \prod_{k=1}^{n} \gamma_k = \prod_{i=1}^{g} [\alpha_i,\, \beta_i],\ \gamma_k^{o_k} = 1,\ 1 \leq k \leq n \rangle,$$

where the elements α_i, β_i, $1 \leq i \leq g$ represent a symplectic basis of the surface underlying Σ, as in Definition 5.1.1, and the γ_k are the homotopy classes of the loops $\widetilde{\gamma}_k$ as in Definition 6.1.1. By abelianizing expression (6.1.1) we obtain the first integral orbifold homology group $\mathrm{H}_1^{\mathrm{orb}}(\Sigma,\, \mathbb{Z})$

$$(6.1.2)\ \langle \{\alpha_i,\, \beta_i\}_{i=1}^{g},\, \{\gamma_k\}_{k=1}^{n} \mid \sum_{k=1}^{n} \gamma_k = 0,\ o_k\, \gamma_k = 0,\ 1 \leq k \leq n \rangle.$$

The torsion subgroup $\mathrm{H}_1^{\mathrm{orb}}(\Sigma,\, \mathbb{Z})_{\mathrm{T}}$ of the first orbifold integral homology group of the orbisurface Σ is generated by the geometric torsion $\{\gamma_k\}_{k=1}^{n}$ with the relations $o_k\, \gamma_k = 0$ and $\sum_{k=1}^{n} \gamma_n = 0$. There are many free subgroups F of the first orbifold homology group for which $\mathrm{H}_1^{\mathrm{orb}}(\Sigma,\, \mathbb{Z}) = F \oplus \mathrm{H}_1^{\mathrm{orb}}(\Sigma,\, \mathbb{Z})_{\mathrm{T}}$. In what follows we will use the definition

$$(6.1.3)\qquad \mathrm{H}_1^{\mathrm{orb}}(\Sigma,\, \mathbb{Z})_{\mathrm{F}} := \mathrm{H}_1^{\mathrm{orb}}(\Sigma,\, \mathbb{Z}) /\, \mathrm{H}_1^{\mathrm{orb}}(\Sigma,\, \mathbb{Z})_{\mathrm{T}},$$

and we call the left hand side of expression (6.1.3), *the free first orbifold homology group of the orbisurface* Σ; such quotient group is isomorphic to the free group on $2g$ generators $\mathbb{Z}^{2g}$, and there is an isomorphism of groups from $\mathrm{H}_1^{\mathrm{orb}}(\Sigma,\, \mathbb{Z})$ onto $\mathrm{H}_1^{\mathrm{orb}}(\Sigma,\, \mathbb{Z})_{\mathrm{F}} \oplus \mathrm{H}_1^{\mathrm{orb}}(\Sigma,\, \mathbb{Z})_{\mathrm{T}}$. As in (5.1.2), there is a natural *intersection form*

$$(6.1.4)\qquad \cap_{\mathrm{F}}\colon\ \mathrm{H}_1^{\mathrm{orb}}(\Sigma,\, \mathbb{R})_{\mathrm{F}} \otimes \mathrm{H}_1^{\mathrm{orb}}(\Sigma,\, \mathbb{R})_{\mathrm{F}} \to \mathbb{R},$$

which for simplicity we write $\cap_{\mathrm{F}} = \cap$, and a natural isomorphism

$$(6.1.5)\qquad \mathrm{H}_1^{\mathrm{orb}}(\Sigma,\, \mathbb{R})_{\mathrm{F}} \to \mathrm{H}_1(\widehat{\Sigma},\, \mathbb{R})$$

which pullbacks $\cap$ to $\cap = \cap_{\mathrm{F}}$, where $\widehat{\Sigma}$ denotes the underlying surface to Σ.

Example 6.1.4. Let Σ be a compact, connected, orientable smooth orbisurface with underlying topological space equal to a 2-dimensional torus $(\mathbb{R}/\mathbb{Z})^2$. Suppose that Σ has precisely one cone point p_1 of order $o_1 = 2$. Let $\widetilde{\gamma}$ be obtained as boundary loop of closed small disk containing the singular point p_1, and let $\gamma = [\widetilde{\gamma}]$. Let α, β be representative of the standard basis of loops of the surface underlying Σ (recall that α, β are a basis of the quotient free first orbifold homology group of Σ). Then for any point $x_0 \in \Sigma$

$$\pi_1^{\mathrm{orb}}(\Sigma,\, x_0) = \langle \alpha,\, \beta,\, \gamma \,|\, [\alpha,\, \beta] = \gamma,\, \gamma^2 = 1 \rangle,$$

and

$$\mathrm{H}_1^{\mathrm{orb}}(\Sigma,\, \mathbb{Z}) = \langle \alpha,\, \beta,\, \gamma \,|\, \gamma = 1,\, 2\,\gamma = 0 \rangle \simeq \langle \alpha,\, \beta \rangle.$$

6.2. Geometric isomorphisms

An orbifold diffeomorphism between orbisurfaces induces an isomorphism at the level of orbifold fundamental groups, and at the level of first orbifold homology groups.

Definition 6.2.1. *Let $\mathcal{O}$, $\mathcal{O}'$ be compact, connected, orientable smooth orbisurfaces. An isomorphism* $\mathrm{H}_1^{\mathrm{orb}}(\mathcal{O}, \mathbb{Z}) \to \mathrm{H}_1^{\mathrm{orb}}(\mathcal{O}', \mathbb{Z})$ *is* orbisurface geometric *if there exists an orbifold diffeomorphism $\mathcal{O} \to \mathcal{O}'$ which induces it.*

Example 6.2.2. If (M, σ) is a compact and connected symplectic manifold of dimension $2n$, and if T is a $(2n-2)$-dimensional torus which acts freely on (M, σ) by means of symplectomorphisms and whose T-orbits are $(2n-2)$-dimensional symplectic submanifolds of (M, σ), then the torsion part of the first integral orbifold homology group of the surface M/T is trivial. But if the action of the torus is not free then M/T is an orbisurface, and the torsion subgroup is frequently non-trivial, although in a few cases it is trivial. For example:

a) if there is only one singular point in M/T, or
b) if there are precisely two cone points of orders 2 and 3 in M/T, or in general of orders k and $k+1$, for a positive integer k. In this case it is easy to see that the torsion subgroup of the first integral orbifold homology group is trivial because it is generated by γ_1, γ_2 with the relations $k\,\gamma_1 = (k+1)\,\gamma_2 = 0$, and $\gamma_1 + \gamma_2 = 0$, or
c) if there are precisely three cone points of orders 3, 4, 5 in M/T, or more generally any three points whose orders are coprime.

However, in cases a) and b) the orbifolds are not good, so they do not arise as M/T, c.f. Lemma 3.4.1. On the other hand, in case c) the orbifold is good, and as we will prove later it does arise as the orbit space of many symplectic manifolds (M, σ). In this case it is possible to give a description of the monodromy invariant which is analogous to the free case done in Chapter 5.

Recall that, by Lemma 5.1.4, if Σ, Σ' are compact, connected, oriented, smooth orbisurfaces of the same Fuchsian signature, and if $f\colon\ \mathrm{H}_1^{\mathrm{orb}}(\Sigma, \mathbb{Z}) \to \mathrm{H}_1^{\mathrm{orb}}(\Sigma', \mathbb{Z})$ is a group isomorphism for which there exists an orientation preserving orbifold diffeomorphism $i\colon \Sigma \to \Sigma'$ such that $f = i_*$, then f preserves the intersection form.

Example 6.2.3. Let $\mathcal{O}$ be any orbifold with two cone points of orders 10 and 15. The torsion part of the orbifold homology is isomorphic to the quotient of the additive group $\mathbb{Z}_{10} \oplus \mathbb{Z}_{15}$ by the sum of the two obvious generators. This group is isomorphic to $\mathbb{Z}_5$. Take an isomorphism of $\mathbb{Z}_5$, which squares each element. This isomorphism cannot be realized by an orbifold diffeomorphism. This is the case because an orbifold diffeomorphism has to be multiplication by 1 or -1 on each of $\mathbb{Z}_{10}$ and $\mathbb{Z}_{15}$, so it has to be multiplication by 1 or -1 on the quotient $\mathbb{Z}_5$.

Not every automorphism of the first orbifold homology group preserves the order of the orbifold singularities.

EXAMPLE 6.2.4. Let $\mathcal{O}$ be a compact, connected, orientable smooth orbisurface with underlying topological space equal to a 2-dimensional torus. Suppose that $\mathcal{O}$ has precisely two cone points p_1, p_2 of respective orders $o_1 = 5$ and $o_2 = 10$. Then for any point $x_0 \in \mathcal{O}$

$$\pi_1^{\text{orb}}(\mathcal{O},\, x_0) = \langle \alpha,\, \beta,\, \gamma_1,\, \gamma_2 \,|\, \gamma_1\, \gamma_2 = [\alpha,\, \beta],\ \gamma_1^5 = \gamma_2^{10} = 1\rangle,$$

and

$$\mathrm{H}_1^{\text{orb}}(\mathcal{O},\, \mathbb{Z}) = \langle \alpha,\, \beta,\, \gamma_1,\, \gamma_2 \,|\, \gamma_1 + \gamma_2 = 0,\, 5\,\gamma_1 = 10\;\gamma_2 = 0\rangle.$$

The assignment

$$F\colon \alpha \to \alpha,\, \beta \to \beta,\, \gamma_1 \mapsto \gamma_2,\, \gamma_2 \mapsto \gamma_1 \tag{6.2.1}$$

defines a group automorphism of $\mathrm{H}_1^{\text{orb}}(\mathcal{O},\, \mathbb{Z})$. If this automorphism is induced by a diffeomorphism, then the same map on $\pi_1^{\text{orb}}(\mathcal{O},\, x_0)$ should be an isomorphism, which is false since in $\pi_1^{\text{orb}}(\mathcal{O},\, x_0)$ the classes γ_1 and γ_2 have different orders. Thus the assignment (6.2.1) is not geometric.

6.3. Symplectic and torsion geometric maps

Next we introduce the notion of symplectic isomorphism as well as that of singularity-order preserving isomorphism.

Let K, L be arbitrary groups, and let $h\colon K \to L$ be a group isomorphism. We denote by K_{T}, L_{T} the corresponding torsion subgroups, by K_{F}, L_{F} the quotients K/K_{T}, L/L_{T}, by h_{T} the restriction of h to $K_{\mathrm{T}} \to L_{\mathrm{T}}$, and by h_{F} the map $K_{\mathrm{F}} \to L_{\mathrm{F}}$ induced by h.

DEFINITION 6.3.1. *Let $\mathcal{O}$, $\mathcal{O}'$ be compact, connected, orientable smooth orbisurfaces. An isomorphism $Z = \mathrm{H}_1^{\text{orb}}(\mathcal{O},\, \mathbb{Z}) \to Z' = \mathrm{H}_1^{\text{orb}}(\mathcal{O}',\, \mathbb{Z})$ is* torsion geometric *if the isomorphism $Z_{\mathrm{T}} \to Z'_{\mathrm{T}}$ sends a geometric torsion basis to a geometric torsion basis preserving the order of the orbifold singularities.*

DEFINITION 6.3.2. *Let $\mathcal{O}$, $\mathcal{O}'$ be compact, connected, orientable smooth orbisurfaces. An isomorphism $Z = \mathrm{H}_1^{\text{orb}}(\mathcal{O},\, \mathbb{Z}) \to Z' = \mathrm{H}_1^{\text{orb}}(\mathcal{O}',\, \mathbb{Z})$ is* symplectic *if the isomorphism $Z_{\mathrm{F}} \to Z'_{\mathrm{F}}$ respects the symplectic form, c.f. (6.1.4), i.e. the matrix of the isomorphism w.r.t. symplectic bases is in the integer symplectic group.*

DEFINITION 6.3.3. *Let Z, Z' be respectively the first integral orbifold homology groups of compact connected orbisurfaces $\mathcal{O}$, $\mathcal{O}'$ of the same Fuchsian signature. We define the following set of isomorphisms*

$$\mathrm{S}^{Z,Z'} := \{f \in \mathrm{Iso}(Z,\, Z') \,|\, f \text{ is torsion geometric}\}, \tag{6.3.1}$$

and

$$\mathrm{Sp}(Z,\, Z') := \{h \in \mathrm{Iso}(Z,\, Z') \,|\, h \text{ is symplectic}\}. \tag{6.3.2}$$

DEFINITION 6.3.4. *Let Z_1, Z_2 be respectively the first integral orbifold homology groups of compact connected orbisurfaces $\mathcal{O}_1$, $\mathcal{O}_2$ of the same Fuchsian signature. Let $f_i\colon Z_i \to T$ be homomorphisms into a torus T. We say that f_1 is* $\mathrm{Sp}(Z_1, Z_2) \cap \mathrm{S}^{Z_1,Z_2}$-equivalent *to f_2 if there exists an isomorphism $i\colon Z_1 \to Z_2$ such that there is an identity of maps $f_2 = f_1 \circ i$ and $i \in \mathrm{Sp}(Z_1, Z_2) \cap \mathrm{S}^{Z_1,Z_2}$.*

6.4. Geometric isomorphisms: characterization

The main result of this section is Theorem 6.4.2, where we give a characterization of geometric isomorphisms in terms of symplectic maps, c.f. Definition 6.3.2, and torsion geometric maps, c.f. Definition 6.3.1. Then we deduce from it Proposition 7.1.1 which is a key ingredient of the proof of the classification theorem.

LEMMA 6.4.1. *Let Σ_1, Σ_2 be compact, connected, oriented smooth orbisurfaces with the same Fuchsian signature $(g; \vec{o})$, and suppose that G is an isomorphism from the first integral orbifold homology group $\mathrm{H}_1^{\mathrm{orb}}(\Sigma_1, \mathbb{Z})$ onto $\mathrm{H}_1^{\mathrm{orb}}(\Sigma_2, \mathbb{Z})$ which is symplectic and torsion geometric, c.f. Definition 6.3.2 and Definition 6.3.1. Then there exists an orbifold diffeomorphism $g\colon \Sigma_1 \to \Sigma_2$ such that $G_{\mathrm{T}} = (g_*)_{\mathrm{T}}$ and $G_{\mathrm{F}} = (g_*)_{\mathrm{F}}$.*

PROOF. Because an orbifold is classified up to orbifold diffeomorphisms by its Fuchsian signature, c.f. Theorem 9.5.2, without loss of generality we may assume that $\Sigma = \Sigma_1 = \Sigma_2$, and let $\{\gamma_k\}$ be a geometric torsion basis of $\mathrm{H}_1^{\mathrm{orb}}(\Sigma, \mathbb{Z})$, c.f Definition 6.1.1. Choose an orbifold atlas for Σ such that the orbifold chart U_k around the k^{th} singular point p_k is homeomorphic to a disk D_k modulo a finite group of diffeomorphisms, and such that every singular point is contained in precisely one chart. The oriented boundary loop ∂U_k represents the class $\gamma_k \in \mathrm{H}_1^{\mathrm{orb}}(\Sigma, \mathbb{Z})_{\mathrm{T}}$. This in particular implies that there exists an orbifold diffeomorphism $f_\tau\colon U := \bigcup_k U_k \to U$ such that $f_\tau(p_k) = p_{\tau(k)}$ and

$$f_\tau(U_k) = U_{\tau(k)}. \tag{6.4.1}$$

Replace each orbifold chart around a singular point of Σ by a manifold chart. This gives rise to a manifold atlas, which defines a compact, connected, smooth orientable surface $\widehat{\Sigma}$ without boundary[2] with the same underlying space as that of the orbisurface Σ. Let $\mathrm{H}_1^{\mathrm{orb}}(\Sigma, \mathbb{Z})_{\mathrm{F}} \to \mathrm{H}_1(\widehat{\Sigma}, \mathbb{Z})$ be the natural intersection form preserving automorphism in (6.1.5). The automorphism obtained from G_{F} by conjugation with the aforementioned automorphism preserves the intersection form on $\mathrm{H}_1(\widehat{\Sigma}, \mathbb{Z})$. By Lemma 5.1.4 there is a surface diffeomorphism which induces it, which sends U_k to $U_{\tau(k)}$. Because each diffeomorphism of a circle is isotopic to a rotation or a reflection, by (6.4.1) the restriction of the aforementioned surface diffeomorphism

[2]recall that we are always assuming, unless otherwise specified, that all manifolds and orbifolds in this paper have no boundary.

to the punctured surface $\widehat{\Sigma} \setminus \bigcup_k U_k$ may be glued to f_τ, along the boundary circles ∂U_k, to give rise to an orbifold diffeomorphism of Σ which satisfies the required properties. $\square$

THEOREM 6.4.2. *Let Σ_1, Σ_2 be compact, connected, orientable smooth orbisurfaces of the same Fuchsian signature. An isomorphism $\mathrm{H}_1^{\mathrm{orb}}(\Sigma_1, \mathbb{Z}) \to \mathrm{H}_1^{\mathrm{orb}}(\Sigma_2, \mathbb{Z})$ is orbisurface geometric if and only if it is symplectic and torsion geometric.*

PROOF. Suppose that $G\colon Z_1 := \mathrm{H}_1^{\mathrm{orb}}(\Sigma_1, \mathbb{Z}) \to Z_2 := \mathrm{H}_1^{\mathrm{orb}}(\Sigma_2, \mathbb{Z})$ is an orbisurface geometric isomorphism, c.f. Definition 6.2.1. It follows from the definition of orbifold diffeomorphism that if the group isomorphism G is induced by an orbifold diffeomorphism, then the induced map G_{F} on the free quotient preserves the intersection form and G sends geometric torsion basis to geometric torsion basis of the orbifold homology preserving the order of the orbifold singularities.

Conversely, suppose that G is symplectic and torsion geometric, c.f. Definition 6.3.2, Definition 6.3.1. By Theorem 9.5.2 we may assume without loss of generality that $\Sigma = \Sigma_1 = \Sigma_2$, so G is an automorphism of $\mathrm{H}_1^{\mathrm{orb}}(\Sigma, \mathbb{Z})$. By Lemma 6.4.1 there exists an orbifold diffeomorphism $g\colon \Sigma \to \Sigma$ such that

$$(g_*)_{\mathrm{F}} = G_{\mathrm{F}}, \quad (g_*)_{\mathrm{T}} = G_{\mathrm{T}}. \tag{6.4.2}$$

Let us define the mapping

$$K := g_* \circ G^{-1}\colon\ \mathrm{H}_1^{\mathrm{orb}}(\Sigma, \mathbb{Z}) \to \mathrm{H}_1^{\mathrm{orb}}(\Sigma, \mathbb{Z}). \tag{6.4.3}$$

K given by (6.4.3) is a group isomorphism because it is the composite of two group isomorphisms. Because of the identities in expression (6.4.2), K satisfies that

$$K_{\mathrm{T}} = \mathrm{Id}_{\mathrm{H}_1^{\mathrm{orb}}(\Sigma, \mathbb{Z})_{\mathrm{T}}}, \quad K_{\mathrm{F}} = \mathrm{Id}_{\mathrm{H}_1^{\mathrm{orb}}(\Sigma, \mathbb{Z})_{\mathrm{F}}}. \tag{6.4.4}$$

If the genus of the underlying surface $|\Sigma|$ is 0, then $\mathrm{H}_1^{\mathrm{orb}}(\Sigma, \mathbb{Z})_{\mathrm{F}}$ is trivial and $\mathrm{H}_1^{\mathrm{orb}}(\Sigma, \mathbb{Z})_{\mathrm{T}} = \mathrm{H}_1^{\mathrm{orb}}(\Sigma, \mathbb{Z})$, $K = K_{\mathrm{T}} = \mathrm{Id}$, so $g_* = G$, and we are done. If the genus of $|\Sigma|$ is strictly positive, then there are two cases. First of all, if K is the identity map, then $g_* = G$, and we are done. If otherwise K is not the identity map, then $g_* \neq G$, and let us choose a free subgroup F of $\mathrm{H}_1^{\mathrm{orb}}(\Sigma, \mathbb{Z})$ such that $\mathrm{H}_1^{\mathrm{orb}}(\Sigma, \mathbb{Z}) = F \oplus \mathrm{H}_1^{\mathrm{orb}}(\Sigma, \mathbb{Z})_{\mathrm{T}}$, and a symplectic basis α_i, β_i of the free group F. To make the forthcoming notation simpler rename $e_{2i-1} = \alpha_i$ and $e_{2i} = \beta_i$, for all i such that $1 \leq i \leq g$. Let $\{\gamma_k\}$ be as in Definition 6.1.1. By the right hand side of equation (6.4.4), we have that there exist non-negative integers a_k^i such that if n is the number singular of points of Σ, then

$$K(e_i) = e_i + \sum_{k=1}^{n} a_k^i\, \gamma_k. \tag{6.4.5}$$

By the left hand side of (6.4.4),

$$K(\gamma_k) = \gamma_k. \tag{6.4.6}$$

Next we define a new isomorphism, which we call K_{new}, by altering K in the following fashion.

Choose an orbifold atlas for the orbisurface Σ such that for each singular point p_j of Σ there is a unique orbifold chart which contains it, and which is homeomorphic to a disk D_j modulo a finite group of diffeomorphisms. Let $\widehat{\Sigma}$ be the smooth surface whose underlying topological space is the underlying surface $|\Sigma|$ to Σ, and whose smooth structure is given by the atlas defined by replacing each chart which contains a singular point by a manifold chart from the corresponding disk. Let $\widetilde{e}_i$ be a loop which represents the homology class $e_i \in \mathrm{H}_1^{\mathrm{orb}}(\Sigma, \mathbb{Z})$. Take an annulus $\mathcal{A}$ in Σ such that the loop $\widetilde{e}_i$ crosses $\mathcal{A}$ exactly once, in the sense that it intersects the boundary $\partial\mathcal{A}$ of $\mathcal{A}$ exactly twice, once at each of the two boundary components of $\partial\mathcal{A}$, and such that it contains the j^{th} singular point p_j of Σ which is enclosed by the oriented loop whose homology class is γ_j, and no other singular point of Σ. This can be done by choosing the annulus $\mathcal{A}$ in such a way that its boundary curves represent the same class as the dual element e_{i+1} to e_i (instead of e_{i+1} we may have e_{i-1}, depending on how the symplectic basis of e_i is arranged). Equip the topological space $|\mathcal{A}| \subset |\Sigma| = |\widehat{\Sigma}|$ with the smooth structure which comes from restricting to $|\mathcal{A}|$ the charts of the smooth structure of $\widehat{\Sigma}$, and let $\widehat{\mathcal{A}}$ be the smooth submanifold-with-boundary of $\widehat{\Sigma}$ which arises in such way. The loop $\widetilde{e}_i$ intersects the boundary of the annulus at an initial point x_{ij} and at an end point y_{ij}, and intersects the annulus itself at a curved segment path $[x_{ij}, y_{ij}]$, which by possibly choosing a different representative of the class e_i, may be assumed to be a smooth embedded 1-dimensional submanifold-with-boundary of $\widehat{\mathcal{A}}$. Replace the segment $[x_{ij}, y_{ij}]$ of the loop $\widetilde{e}_i$ by a smoothly embedded path $P(x_{ij}, y_{ij})$ which starts at the point x_{ij}, goes towards the cone point enclosed by γ_j while inside of the aforementioned annulus, and goes around it precisely once, to finally come back to end up at the end point y_{ij}. The replacement of the segment $[x_{ij}, y_{ij}]$ by the path $P(x_{ij}, y_{ij})$ gives rise to a new loop $\widetilde{e}'_i$, which agrees with the loop $\widetilde{e}_i$ outside of $\mathcal{A}$.

We claim that there exists an orbifold diffeomorphism $\widehat{h}_{ij} \colon \widehat{\Sigma} \to \widehat{\Sigma}$ which is the identity map outside of the annulus and which sends the segment $[x_{ij}, y_{ij}]$ to the path $P(x_{ij}, y_{ij})$, and hence the loop $\widetilde{e}_i$ to the loop $\widetilde{e}'_i$. Indeed, let $\widehat{f}$ be a diffeomorphism of the aforementioned annulus $\widehat{\mathcal{A}}$ with

$$\widehat{f}([x_{ij}, y_{ij}]) = P(x_{ij}, y_{ij}),$$

and with $\widehat{f}$ being the identity on $\partial\widehat{\mathcal{A}}$. Then let $\widehat{k} \colon \widehat{\mathcal{A}} \to \widehat{\mathcal{A}}$ be a diffeomorphism which is the identity on $P(x_{ij}, y_{ij})$ and on $\partial\widehat{\mathcal{A}}$, such that $\widehat{k}(\widehat{f}(p_j)) = p_j$. Such a diffeomorphism $\widehat{k}$ exists since cutting the annulus along $P(x_{ij}, y_{ij})$ gives a disk, and in a disk there is a diffeomorphism taking any interior point x to any other interior point y and fixing a neighborhood of the boundary. The composition $\widehat{h}'_{ij} := \widehat{k} \circ \widehat{f} \colon \widehat{\mathcal{A}} \to \widehat{\mathcal{A}}$ is a diffeomorphism which is the identity on $\partial\widehat{\mathcal{A}}$, which sends the segment $[x_{ij}, y_{ij}]$ to the path

$P(x_{ij}, y_{ij})$ and such that

$$\widehat{h}'_{ij}(p_j) = p_j. \tag{6.4.7}$$

Because the mapping $\widehat{h}'_{ij}$ is the identity along $\partial\widehat{\mathcal{A}}$, it extends to a diffeomorphism $\widehat{h}_{ij}$ along such boundary, which satisfies the required properties. Since p_j is contained in a unique orbifold chart and (6.4.7) holds, the way in which we defined the smooth structure of $\widehat{\Sigma}$ from the orbifold structure of Σ gives that $\widehat{h}_{ij}$ defines an orbifold diffeomorphism $\Sigma \to \Sigma$. To emphasize that $\widehat{h}_{ij}$ is a diffeomorphism at the level of orbifolds, we denote it by $h_{ij} \colon \Sigma \to \Sigma$.

The isomorphism h^*_{ij} induced on the orbifold homology by the orbifold diffeomorphism h_{ij} is given by

$$h^*_{ij}(e_k) = e_k,\ k \neq i, \quad h^*_{ij}(e_i) = e_i + \gamma_j, \tag{6.4.8}$$

$$h^*_{ij}(\gamma_k) = \gamma_k, \tag{6.4.9}$$

where $1 \leq i \leq 2g$ and $1 \leq k \leq n$, and notice that in the first equality we have used that the boundary curve of the annulus is in the class of e_{i+1} (for brevity we are writing h^*_{ij} instead of $(h_{ij})_*$). Define $f_{ij} \colon \Sigma \to \Sigma$ to be the orbifold diffeomorphism obtained by composing h_{ij} with itself precisely a^j_i times. It follows from (6.4.8) and (6.4.9) that

$$f^*_{ij}(e_k) = e_k \ \text{ if } k \neq i,\ \ f^*_{ij}(e_i) = e_i + a^j_i\, \gamma_j, \tag{6.4.10}$$

and

$$f^*_{ij}(\gamma_k) = \gamma_k, \tag{6.4.11}$$

where $1 \leq i \leq 2g$ and $1 \leq k \leq n$.

The isomorphisms f^*_{ij} commute with each other, because they are the identity on the torsion subgroup, and only change one loop of the free part which does not affect the other loops[3]. Therefore combining expressions (6.4.5), (6.4.10) and (6.4.11) we arrive at the identity

$$((f^*_{ij})^{-1} \circ K)(e_i) = e_i + \sum_{k=1,\, k\neq j}^{n} a^i_k\, \gamma_k. \tag{6.4.12}$$

On the other hand, it follows from (6.4.6) and (6.4.11) that

$$((f^*_{ij})^{-1} \circ K)(\gamma_k) = \gamma_k. \tag{6.4.13}$$

Now consider the isomorphism K_{new} of the orbifold homology group defined by

$$K_{\mathrm{new}} := \big(\bigcirc_{1\leq i\leq 2g,\, 1\leq j\leq n}\, (f^*_{ij})^{-1}\big) \circ K, \tag{6.4.14}$$

where recall that in (6.4.14), g is the genus of the surface underlying Σ, and n is the number of singular points. It follows from expression (6.4.13), and from (6.4.12), by induction on i and j, that K_{new} given by (6.4.14) is

[3] Observe that this is false in the fundamental group.

the identity map on the orbifold homology, which then by formula (6.4.3) implies that

$$\begin{aligned} G = g_* \circ K^{-1} &= g_* \circ (\bigcirc_{1\leq i\leq 2g,\, 1\leq j\leq n} f_{ij}^*) \\ &= (g \circ (\bigcirc_{1\leq i\leq 2g,\, 1\leq j\leq n} f_{ij}))_*, \end{aligned}$$

and hence G is a geometric isomorphism induced by

$$g \circ (\bigcirc_{1\leq i\leq 2g,\, 1\leq j\leq n} f_{ij}),$$

which is a composite of orbifold diffeomorphisms, and hence an orbifold diffeomorhism itself. $\square$

CHAPTER 7

Classification

This chapter extends the results of Chapter 5 to non-free actions.

Some of the statements and proofs in the present chapter are analogous to those of Chapter 5, and we do not repeat them.

7.1. Monodromy invariant

We define the Fuchsian signature monodromy space, whose elements give one of the ingredients of the classification theorems.

Recall that the mappings μ_{h}, μ'_{h} are, respectively, the homomorphisms induced on homology by the monodromy homomorphisms μ, μ' of the connections Ω, Ω' of symplectically orthogonal complements to the tangent spaces to the T-orbits in M, M', respectively (c.f. Proposition 2.4.1). Recall that ν, ν' are the unique 2-forms respectively on M/T and M'/T such that $\pi^*\nu|_{\Omega_x} = \sigma|_{\Omega_x}$ and $\pi'^*\nu'|_{\Omega'_{x'}} = \sigma'|_{\Omega'_{x'}}$, for every $x \in M$, $x' \in M'$, c.f. Lemma 3.2.1.

Proposition 7.1.1. *Let (M, σ) and (M', σ') be two compact connected $2n$-dimensional symplectic manifolds equipped with an effective symplectic action of a $(2n-2)$-dimensional torus T for which at least one, and hence every T-orbit is a $(2n-2)$-dimensional symplectic submanifold of (M, σ) and (M', σ'), respectively. Let $K = \mathrm{H}_1^{\mathrm{orb}}(M/T, \mathbb{Z})$ and similarly K'. Suppose that the orbit spaces $(M/T, \nu)$ and $(M'/T, \nu')$ are orbifold symplectomorphic and that μ_{h} is $\mathrm{Sp}(K', K) \cap \mathrm{S}^{K', K}$-equivalent to μ'_{h} via an automorphism G from the orbifold homology group $\mathrm{H}_1^{\mathrm{orb}}(M'/T, \mathbb{Z})$ onto $\mathrm{H}_1^{\mathrm{orb}}(M/T, \mathbb{Z})$. Then there exists an orbifold diffeomorphism $g\colon M'/T \to M/T$ such that $G = g_*$ and $\mu'_{\mathrm{h}} = \mu_{\mathrm{h}} \circ g_*$.*

Proof. It follows from Theorem 6.4.2 applied to the groups Z_1, Z_2, which respectively are the first integral orbifold homology group of Σ_1, and of Σ_2, where $\Sigma_1 = M'/T$, $\Sigma_2 = M/T$. □

7.1.1. Fuchsian signature space. We define the invariant of (M, σ) which encodes the monodromy of the connection for $\pi\colon M \to M/T$ of symplectic orthogonal complements to the tangent spaces to the T-orbits, c.f. Definition 7.1.6.

Definition 7.1.2. *Let $\mathcal{O}$ be a smooth orbisurface with n cone points p_k, $1 \leq k \leq n$. The* Fuchsian signature $\mathrm{sig}(\mathcal{O})$ of $\mathcal{O}$ *is the $(n+1)$-tuple $(g; \vec{o})$ where g is the genus of the underlying surface to the orbisurface $\mathcal{O}$,*

o_k is the order of the point p_k, which we require to be strictly positive, and $\vec{o} = (o_k)_{k=1}^n$, where $o_k \leq o_{k+1}$, for all $1 \leq k \leq n-1$.

DEFINITION 7.1.3. *Let $\vec{o}$ be an n-dimensional tuple of strictly positive integers. We define*

$$\mathcal{M}\mathrm{S}_n^{\vec{o}} := \{B \in \mathrm{GL}(n,\, \mathbb{Z}) \,|\, B \cdot \vec{o} = \vec{o}\},$$

i.e. $\mathcal{M}\mathrm{S}_n^{\vec{o}}$ is the group of n-dimensional matrices which permute elements preserving the tuple of orders $\vec{o}$.

DEFINITION 7.1.4. *Let $(g;\, \vec{o})$ be an $(n+1)$-tuple of integers, where the o_k's are strictly positive and non-decreasingly ordered. Let T be a torus. Let $\mathcal{G}_{(g,\,\vec{o})}$ be the group of matrices*

$$\{\begin{pmatrix} A & 0 \\ C & D \end{pmatrix} \in \mathrm{GL}(2g+n,\, \mathbb{Z}) \,|\, A \in \mathrm{Sp}(2g,\, \mathbb{Z}),\, D \in \mathcal{M}\mathrm{S}_n^{\vec{o}}\}, \tag{7.1.1}$$

where $\mathrm{Sp}(2g,\, \mathbb{Z})$ is the group of $2g$-dimensional symplectic matrices with integer entries, c.f. Section 5.1.1, and $\mathcal{M}\mathrm{S}_n^{\vec{o}}$ is the group of n-dimensional matrices which permute elements preserving the tuple of orders $\vec{o}$, c.f. Definition 7.1.3. The Fuchsian signature space associated to $(g;\, \vec{o})$ *is the quotient space*

$$T^{2g+n}_{(g;\,\vec{o})}/\mathcal{G}_{(g,\,\vec{o})} \tag{7.1.2}$$

where $T^{2g+n}_{(g;\,\vec{o})}$ is

$$\{(t_i)_{i=1}^{2g+m} \in T^{2g+m} \,|\, \prod_{i=2g+1}^{2g+m} t_i = 1 \,\text{and the order of}\; t_i \;\text{is}\; o_i,\, 2g{+}1 \leq i \leq 2g{+}m\,\},$$

and $\mathcal{G}_{(g;\,\vec{o})}$ acts on $T^{2g+n}_{(g;\,\vec{o})}$ as in Definition 5.1.5.

REMARK 7.1.5. The lower left block C in the definition of $\mathcal{G}_{(g;\,\vec{o})}$ in Definition 7.1.4 of the description of item 4) is allowed to be any matrix; this reflects that there are many free subgroups of the first integral orbifold homology group with together with the torsion span the entire group.

DEFINITION 7.1.6. *Let $(M,\, \sigma)$ be a compact connected $2n$-dimensional symplectic manifold and let T be a $(2n-2)$-dimensional torus which acts effectively on $(M,\, \sigma)$ by means of symplectomorphisms. We furthermore assume that at least one, and hence every T-orbit is $(2n-2)$-dimensional symplectic submanifold of $(M,\, \sigma)$. Let $(g;\, \vec{o}) \in \mathbb{Z}^{1+m}$ be the Fuchsian signature of the orbit space M/T. Let $\{\gamma_k\}_{k=1}^m$ be a geometric torsion basis, c.f. Definition 6.1.1. Let $\{\alpha_i,\, \beta_i\}_{i=1}^g$ be a symplectic basis of a free subgroup of our choice of $\mathrm{H}_1^{\mathrm{orb}}(M/T,\, \mathbb{Z})$, c.f. expression (5.1.3) and* [**41**, Th. 2.3] *whose direct sum with the torsion subgroup is equal to $\mathrm{H}_1^{\mathrm{orb}}(M/T,\, \mathbb{Z})$. Let μ_{h} be the homomomorphism induced on homology by the monodromy homomorphism*

μ associated to the connection Ω, c.f. Proposition 2.4.1. The monodromy invariant of (M, σ, T) *is the $\mathcal{G}_{(g;\,\vec{o})}$-orbit*

$$\mathcal{G}_{(g,\,\vec{o})} \cdot ((\mu_{\mathrm{h}}(\alpha_i),\, \mu_{\mathrm{h}}(\beta_i))_{i=1}^{g},\, (\mu_{\mathrm{h}}(\gamma_k))_{k=1}^{m}), \tag{7.1.3}$$

of the $(2g+n)$-tuple $((\mu_{\mathrm{h}}(\alpha_i),\, \mu_{\mathrm{h}}(\beta_i))_{i=1}^{g},\, (\mu_{\mathrm{h}}(\gamma_k))_{k=1}^{n})$, where $\mathcal{G}_{(g,\,\vec{o})}$ is the group of matrices given in (7.1.1).

Because the invariant in Definition 7.1.6 depends on choices, it is unclear whether it is well defined.

LEMMA 7.1.7. *Let $(M,\, \sigma)$ be a compact connected $2n$-dimensional symplectic manifold and let T be a $(2n-2)$-dimensional torus which acts effectively on $(M,\, \sigma)$ by means of symplectomorphisms. We furthermore assume that at least one, and hence every T-orbit is $(2n-2)$-dimensional symplectic submanifold of $(M,\, \sigma)$. The monodromy invariant of $(M,\, \sigma,\, T)$ is well defined, in the following sense.*

Suppose that the signature of M/T is $(g;\, \vec{o}) \in \mathbb{Z}^{1+m}$. Let F, F' be any two free subgroups of $\mathrm{H}_1^{\mathrm{orb}}(M/T,\, \mathbb{Z})$ whose direct sum with the torsion subgroup is equal to $\mathrm{H}_1^{\mathrm{orb}}(M/T,\, \mathbb{Z})$, and let α_i, β_i and α_i', β_i' be symplectic bases of F, F', respectively. Let $\tau \in \mathrm{S}_m^{\vec{o}}$. Let μ_{h} be the homomorphism induced in homology by means of the Hurewicz map from the monodromy homomorphism μ of the connection of symplectic orthogonal complements to the tangent spaces to the T-orbits, c.f. Proposition 2.4.1. Then the tuples $((\mu_{\mathrm{h}}(\alpha_i),\, \mu_{\mathrm{h}}(\beta_i))_{i=1}^{g},\, (\mu_{\mathrm{h}}(\gamma_k))_{k=1}^{m})$ and $((\mu_{\mathrm{h}}(\alpha_i'),\, \mu_{\mathrm{h}}(\beta_i'))_{i=1}^{g},\, (\mu_{\mathrm{h}}(\gamma_{\tau(k)})_{k=1}^{m})$ lie in the same $\mathcal{G}_{(g;\,\vec{o})}$-orbit in the Fuchsian signature space.

PROOF. A symplectic basis of the maximal free subgroup F of the orbifold homology group $\mathrm{H}_1^{\mathrm{orb}}(M/T,\, \mathbb{Z})$ may be taken to a symplectic basis of the maximal free subgroup F' of $\mathrm{H}_1^{\mathrm{orb}}(M/T,\, \mathbb{Z})$ by a matrix

$$X := \begin{pmatrix} A & 0 \\ C & \mathrm{Id} \end{pmatrix} \in \mathrm{GL}(2g+m,\, \mathbb{Z}),$$

where the upper block A is a $2g$-dimensional matrix in the integer symplectic linear group $\mathrm{Sp}(2g,\, \mathbb{Z})$, and the lower block C is $(m \times 2g)$-dimensional matrix with integer entries. Here Id denotes the n-dimensional identity matrix, and 0 is the $(m \times 2g)$-dimensional matrix all the entries of which equal 0. A geometric torsion basis can be taken to another geometric torsion basis by preserving the order $\vec{o}$ of the orbifold singularities by a matrix of the form

$$Y := \begin{pmatrix} \mathrm{Id} & 0 \\ 0 & N \end{pmatrix} \in \mathrm{GL}(2g+m,\, \mathbb{Z}),$$

for a certain matrix $N \in \mathcal{M}\mathrm{S}_m^{\vec{o}}$, and the product matrix

$$X\,Y = \begin{pmatrix} A & 0 \\ C & N \end{pmatrix}$$

lies in $\mathcal{G}_{(g;\,\vec{o})}$. □

With this matrix terminology, we can restate Proposition 7.1.1 in the following terms.

See the paragraph preceding Proposition 7.1.1 for a reminder of the terminology which we use next.

PROPOSITION 7.1.8. *Let* $(M,\, \sigma)$ *and* $(M',\, \sigma')$ *be two compact connected* $2n$*-dimensional symplectic manifolds equipped with an effective symplectic action of a* $(2n-2)$*-dimensional torus* T *for which at least one, and hence every* T*-orbit is a* $(2n-2)$*-dimensional symplectic submanifold of* $(M,\, \sigma)$ *and* $(M',\, \sigma')$*, respectively. Suppose that the symplectic orbit spaces* $(M/T,\, \nu)$ *and* $(M'/T,\, \nu')$ *have Fuchsian signature* $(g;\, \vec{o})$*, that they are orbifold symplectomorphic and that the* $\mathcal{G}_{(g;\, \vec{o})}$*-orbits of the* $(2g+m)$*-tuples of elements* $((\mu_{\mathrm{h}}(\alpha_i'),\, \mu_{\mathrm{h}}(\beta_i'))_{i=1}^{g},\, (\mu_{\mathrm{h}}(\gamma_k'))_{k=1}^{m})$ *and* $((\mu_{\mathrm{h}}'(\alpha_i'),\, \mu_{\mathrm{h}}'(\beta_i'))_{i=1}^{g},\, (\mu_{\mathrm{h}}'(\gamma_k'))_{k=1}^{m})$ *are equal, where* $\alpha_i,\, \beta_i$ *and* $\alpha_i',\, \beta_i'$ *are respectively symplectic bases of free homology subgroups of the groups* $\mathrm{H}_1^{\mathrm{orb}}(M/T,\, \mathbb{Z})$ *and* $\mathrm{H}_1^{\mathrm{orb}}(M'/T,\, \mathbb{Z})$ *which together with the corresponding torsion subgroups span the entire group, and* $\gamma_k,\, \gamma_k'$ *are corresponding geometric torsion bases. Then there exists an orbifold diffeomorphism* $g\colon M'/T \to M/T$ *such that* $\mu_{\mathrm{h}}' = \mu_{\mathrm{h}} \circ g_*$*.*

PROOF. By assumption the $\mathcal{G}_{(g;\, \vec{o})}$-orbits of the $(2g+m)$-tuples $((\mu_{\mathrm{h}}(\alpha_i'),\, \mu_{\mathrm{h}}(\beta_i'))_{i=1}^{g},\, (\mu_{\mathrm{h}}(\gamma_k'))_{k=1}^{m})$ and $((\mu_{\mathrm{h}}'(\alpha_i'),\, \mu_{\mathrm{h}}'(\beta_i'))_{i=1}^{g},\, (\mu_{\mathrm{h}}'(\gamma_k))_{k=1}^{m})$ are equal, and hence $\mu_{\mathrm{h}}' = \mu_{\mathrm{h}} \circ G$, where G is the isomorphism defined by $G(\alpha_i) = \alpha_i'$, $G(\beta_i) = \beta_i'$ and $G(\gamma_i) = \gamma_{\tau(i)}'$. By its definition G is symplectic and torsion geometric, i.e. $G \in \mathrm{Sp}(K',\, K) \cap S^{K',\, K}$, where $K := \mathrm{H}_1^{\mathrm{orb}}(M/T,\, \mathbb{Z})$ and $K' := \mathrm{H}_1^{\mathrm{orb}}(M'/T,\, \mathbb{Z})$. Now the result follows from Proposition 7.1.1. □

7.2. Uniqueness

7.2.1. List of ingredients of $(M,\, \sigma,\, T)$**.** We start by assigning a list of invariants to $(M,\, \sigma)$.

DEFINITION 7.2.1. *Let* $(M,\, \sigma)$ *be a compact connected* $2n$*-dimensional symplectic manifold equipped with an effective symplectic action of a* $(2n-2)$*-*
-dimensional torus T *for which at least one, and hence every* T*-orbit is a* $(2n-2)$*-dimensional symplectic submanifold of* M*. The* list of ingredients *of* $(M,\, \sigma,\, T)$ *consists of the following items.*

1) *The Fuchsian signature* $(g;\, \vec{o}) \in \mathbb{Z}^{1+m}$ *of the orbisurface* M/T *(c.f. Remark 2.3.6 and Definition 7.1.2).*
2) *The total symplectic area of the symplectic orbisurface* $(M/T,\, \nu)$*, where the symplectic form* ν *is defined by the condition* $\pi^*\nu|_{\Omega_x} = \sigma|_{\Omega_x}$ *for every* $x \in M$*, where* $\pi\colon M \to M/T$ *is the projection map and* $\Omega_x = (\mathrm{T}_x(T \cdot x))^{\sigma_x}$ *(c.f. Lemma 3.2.1).*
3) *The unique non-degenerate antisymmetric bilinear form* $\sigma^{\mathfrak{t}}\colon \mathfrak{t} \times \mathfrak{t} \to \mathbb{R}$ *on the Lie algebra* $\mathfrak{t}$ *of* T *such that for all* $X,\, Y \in \mathfrak{t}$ *and all* $x \in M$ $\sigma_x(X_M(x),\, Y_M(x)) = \sigma^{\mathfrak{t}}(X,\, Y)$ *(c.f. Lemma 2.1.1).*

4) *The monodromy invariant of* (M, σ, T)*, i.e. the* $\mathcal{G}_{(g;\vec{o})}$*-orbit*
$$\mathcal{G}_{(g,\vec{o})} \cdot ((\mu_{\rm h}(\alpha_i),\, \mu_{\rm h}(\beta_i))_{i=1}^{g},\, (\mu_{\rm h}(\gamma_k))_{k=1}^{m}),$$
of the $(2g+m)$*-tuple* $((\mu_{\rm h}(\alpha_i),\, \mu_{\rm h}(\beta_i))_{i=1}^{g},\, (\mu_{\rm h}(\gamma_k))_{k=1}^{m})$*, c.f. Definition 7.1.6.*

THEOREM 7.2.2. *Suppose that* G *is the first integral orbifold homology group of a compact, connected, orientable smooth orbisurface of Fuchsian signature* $(g;\, \vec{o}) \in \mathbb{Z}^{1+m}$*. Choose a set of generators* $\{\alpha_i,\, \beta_i\}_{i=1}^{g}$ *of a maximal*[1] *free subgroup and let* $\{\gamma_k\}_{k=1}^{m}$ *be a geometric torsion basis. The group of geometric isomorphisms of* G*, c.f. Definition 6.2.1, is equal to the group of isomorphisms of* G *induced by linear isomorphisms* f_B *of* $\mathbb{Z}^{2g+m}$ *where* $B \in \mathcal{G}_{(g;\vec{o})}$*, and* $\mathcal{G}_{(g;\vec{o})}$ *is given in Definition 7.1.4. (Recall that we had to choose the generators* α_i*,* β_i*,* γ_k *in order to define an endomorphism* f_B *of* $\mathbb{Z}^{2g+m}$ *from the matrix* B*).*

PROOF. After fixing a group, the statement corresponds to that of Theorem 6.4.2 formulated in the language of matrices. □

7.2.2. Uniqueness Statement. We prove that the list of ingredients of (M, σ, T) as in Definition 7.2.1 is a complete set of invariants of (M, σ, T).

LEMMA 7.2.3. *Let* (M, σ) *be a compact connected* $2n$*-dimensional symplectic manifold equipped with an effective symplectic action of a* $(2n-2)$*-dimensional torus* T *for which at least one, and hence every* T*-orbit is a* $(2n-2)$*-dimensional symplectic submanifold of* (M, σ)*. Then if* (M', σ') *is a compact connected* $2n$*-dimensional symplectic manifold equipped with an effective symplectic action of* T *for which at least one, and hence every* T*-orbit is a* $(2n-2)$*-dimensional symplectic submanifold of* (M, σ')*, and* (M', σ') *is* T*-equivariantly symplectomorphic to* (M, σ)*, then the list of ingredients of* (M', σ', T) *is equal to the list of ingredients of* (M, σ, T)*.*

The proof of Lemma 7.2.3 is analogous to the proof of Lemma 5.2.3.

PROPOSITION 7.2.4. *Let* (M, σ) *be a compact connected* $2n$*-dimensional symplectic manifold which is equipped with an effective symplectic action of a* $(2n-2)$*-dimensional torus* T *for which at least one, and hence every* T*-orbit is a* $(2n-2)$*-dimensional symplectic submanifold. Then if* (M', σ') *is a compact connected* $2n$*-dimensional symplectic manifold equipped with an effective symplectic action of* T *for which at least one, and hence every* T*-orbit is a* $(2n-2)$*-dimensional symplectic submanifold of* (M', σ') *and the list of ingredients of* (M', σ', T) *is equal to the list of ingredients of* (M, σ, T)*, then* (M', σ') *is* T*-equivariantly symplectomorphic to* (M, σ)*.*

PROOF. Suppose that the list of ingredients of (M, σ) equals the list of ingredients of (M', σ'). Because M/T and M'/T have the same Fuchsian signature and symplectic area, by the orbifold version of Moser's theorem [**39**, Th. 3.3], the (compact, connected, smooth, orientable) orbisurfaces

[1] This means that together with the torsion subgroup spans the entire group.

$(M/T,\, \nu)$ and $(M'/T,\, \nu')$ are symplectomorphic, where ν and ν' are the symplectic forms given by Lemma 3.2.1.

Let μ, μ', μ_{h}, μ'_{h} be the monodromy homomorphisms from the orbifold fundamental groups $\pi_1^{\mathrm{orb}}(M/T,\, p_0)$, $\pi_1^{\mathrm{orb}}(M'/T,\, p_0)$, and from the first integral orbifold homology groups $\mathrm{H}_1^{\mathrm{orb}}(M/T,\, \mathbb{Z})$, $\mathrm{H}_1^{\mathrm{orb}}(M'/T,\, \mathbb{Z})$ into the torus T, respectively associated to the symplectic manifolds $(M,\, \sigma)$, $(M',\, \sigma')$ as in Definition 4.3.1. Because ingredient 4) of $(M,\, \sigma)$ equals ingredient 4) of $(M',\, \sigma')$, by Proposition 7.1.8 and Definition 6.2.1 there exists an orbifold diffeomorphism $F\colon M/T \to M'/T$ such that $\mu_{\mathrm{h}} = \mu'_{\mathrm{h}} \circ F_*$, and hence $\mu = \mu' \circ F_*$.

The remaining part of the proof is analogous to the proof in the free case, "proof of Proposition 5.2.4", and the details of the arguments that follow may be found there. If $\nu_0 := F^*\nu'$, the symplectic orbit space $(M/T,\, \nu_0)$ is symplectomorphic to $(M'/T,\, \nu')$, by means of F. Let $\widetilde{\nu_0}$ be the pullback of the 2-form ν_0 by the orbifold universal cover $\psi\colon \widetilde{M/T} \to M/T$ of M/T based at $p_0 = \psi(x_0)$, and similarly we define $\widetilde{\nu'}$ by means of the universal cover $\psi'\colon \widetilde{M'/T} \to M'/T$ based at $F(p_0)$. As in the proof of Proposition 5.2.4, the orbifold symplectomorphism F between $(M/T,\, \nu_0)$ and $(M'/T,\, \nu')$ lifts to a unique symplectomorphism $\widetilde{F}$ between $(\widetilde{M/T},\, \widetilde{\nu_0})$ and $(\widetilde{M'/T},\, \widetilde{\nu'})$ such that $\widetilde{F}(x_0) = x'_0$. Then the assignment

$$[[\gamma],\, t]_{\pi_1^{\mathrm{orb}}(M/T,\, p_0)} \mapsto [\widetilde{F}([\gamma]),\, t]_{\pi_1^{\mathrm{orb}}(M'/T,\, p'_0)},$$

is a T-equivariant symplectomorphism between $\widetilde{M/T} \times_{\pi_1^{\mathrm{orb}}(M/T,\, p_0)} T$ and $\widetilde{M'/T} \times_{\pi_1^{\mathrm{orb}}(M'/T,\, p'_0)} T$, which by Theorem 3.4.3, gives rise to a T-equivariant symplectomorphism between $(M,\, \sigma)$ and $(M',\, \sigma')$. □

7.3. Existence

7.3.1. List of ingredients for T. We assign to a torus T a list of four ingredients. This is analogous to Definition 5.3.1.

DEFINITION 7.3.1. *Let T be a torus. The* list of ingredients for T *consists of the following.*

i) *An $(m+1)$-tuple $(g;\, \vec{o})$ of integers, where $\vec{o}$ is non-decreasingly ordered and consists of strictly positive integers and m is a non--negative integer, and such that $(g;\, \vec{o})$ is not of the form $(0;\, o_1)$ or of the form $(0;\, o_1,\, o_2)$ with $o_1 < o_2$.*
ii) *A positive real number $\lambda > 0$.*
iii) *A non-degenerate antisymmetric bilinear form $\sigma^{\mathfrak{t}}\colon \mathfrak{t} \times \mathfrak{t} \to \mathbb{R}$ on the Lie algebra $\mathfrak{t}$ of T.*

iv) *An orbit* $\mathcal{G}_{(g,\vec{o})} \cdot \xi \in T^{2g+m}_{(g;\vec{o})}/\mathcal{G}_{(g,\vec{o})}$ *in the Fuchsian signature space associated to* $(g;\, \vec{o})$, *where* $T^{2g+m}_{(g;\vec{o})}$ *is*

$$\{(t_i)_{i=1}^{2g+m} \in T^{2g+m} \mid \prod_{i=2g+1}^{2g+m} t_i = 1 \,\text{and the order of}\ t_i \ \text{is}\ o_i,\ 2g{+}1 \le i \le 2g{+}m\,\},$$

and where $\mathcal{G}_{(g,\vec{o})}$ *is the group of matrices in Definition 7.1.4.*

REMARK 7.3.2. Assume the notation in Definition 7.3.1. It follows from item iv) that $(g;\, \vec{o})$ is not of the form $(0;\, o_1)$ or of the form $(0;\, o_1,\, o_2)$ with $o_1 < o_2$, which was in turn required in item i); this would not have been necessary, yet by [**58**, Th. 13.3.6] this condition is precisely equivalent to the orbisurface with with Fuchsian signature $(g;\, \vec{o})$ being a good orbisurface, so we felt the condition was meaningful enough to deserve being emphasized.

7.3.2. Existence statement. Any list of ingredients as in Definition 7.3.1 gives rise to one of our manifolds with symplectic T-action. We start with the following observations.

LEMMA 7.3.3. *Let* Σ *be a compact, connected, boundaryless,* 2*-dimensional good orbisurface with* m *singular points of orders* $o_1,\, \ldots,\, o_m$, *and with underlying surface having genus* g. *Let* T *be a torus. Let* $f \colon \pi_1^{\mathrm{orb}}(\Sigma,\, p_0) \to T$ *be a homomorphism, where we write the presentation of* $\pi_1^{\mathrm{orb}}(\Sigma,\, p_0)$ *as in (6.1.1) with* $n = m$. *Consider the diagonal action*

$$\pi_1^{\mathrm{orb}}(\Sigma,\, p_0) \times (\widetilde{\Sigma} \times T) \to (\widetilde{\Sigma} \times T) \tag{7.3.1}$$

given by $x\,(y,\, t) = (x{\star}y^{-1},\ f(x){\cdot}t)$, *where* $\star$ *denotes concatenation of paths. Then the following conditions are equivalent:*

(1) *the action (7.3.1) is free;*
(2) *for each* $k = 1, \ldots, m$ *the order of* $f(\gamma_k)$ *is equal to* o_k*;*
(3) $\ker(f)$ *acts freely on* $\widetilde{\Sigma}$.

PROOF. Statements (1) and (3) are immediately equivalent.

Next we show that (3) implies (2). Let $c_k = f(\gamma_k)$, for each $k = 1, \ldots, m$. Because f is a homomorphism the order of c_k, call it l_k, must divide o_k, and hence $1 \neq l_k < o_k$; hence $(c_k)^{l_k}$ is both in $\ker(f)$ and has fixed points, which contradicts our assumption. The converse follows by a similar reasoning.

To conclude we show that (2) implies (3). Write for simplicity $\Gamma = \pi_1^{\mathrm{orb}}(\Sigma,\, x_0)$, which is acting properly, effectively and smoothly on the smooth surface $\widetilde{\Sigma}$, where Σ is identified with $\widetilde{\Sigma}/\Gamma$ and $\pi \colon \widetilde{\Sigma} \to \Sigma$, $s \mapsto \Gamma \cdot s$ is the canonical projection. Suppose that $\gamma \in \pi_1^{\mathrm{orb}}(\Sigma,\, p_0)$ does not act freely on $\widetilde{\Sigma}$. This means that there exists $s \in \widetilde{\Sigma}$ such that $1 \neq \gamma \in \Gamma_s$. Because the restriction of π to a suitable open neighborhood S_0 of s in $\widetilde{\Sigma}$, together with Γ_s, is an orbifold chart for the open neighborhood $\Sigma_0 = \pi(S_0)$ of the point $x = \pi(s) \in \Sigma$, we conclude that there exists a j such that x is equal to one the singular points x_j of Σ with order o_j. For such a singular point x_j you have an $s_j \in \pi^{-1}(\{x_j\}) \subset \widetilde{\Sigma}$ such that

$$\Gamma_{s_j} = \{\gamma_j^k \mid k \in \mathbb{Z} \setminus o_j\,\mathbb{Z}\},$$

where $o_j \in \mathbb{Z}_{>1}$ is the order of the singularity at x_j.

Now $s \in \pi^{-1}(\{x\}) = \pi^{-1}(\{x_j\})$ together with $s_j \in \pi^{-1}(\{x_j\})$ imply that there exists a $\delta \in \Gamma$ such that $s = \delta\, s_j$, hence

$$\Gamma_s = \Gamma_{\delta\, s_j} = \delta\, \Gamma_{s_j}\, \delta^{-1}.$$

It follows that our $\gamma \in \Gamma_s$ is of the form $\gamma = \delta\, \gamma_j^k\, \delta^{-1}$ for some $k \in \mathbb{Z}$, $k \notin o_j\,\mathbb{Z}$. This implies that

$$f(\gamma) = f(\delta\, \gamma_j^k\, \delta^{-1}) = f(\delta)\, f(\gamma_j)^k\, f(\delta)^{-1} = c_j^k.$$

Here in the second equality we are using that μ is a homomorphism, and in the third equality that T is commutative and $f(\gamma_j) = c_j$. Because we assumed that the order of c_j in T is equal to o_j, the fact that $k \notin o_j\,\mathbb{Z}$ implies that $f(\gamma) \neq 1$, that is, $\gamma \notin \ker(f)$. □

Remark 7.3.4. Assume the terminology of Definition 3.3.1, that M/T is 2-dimensional, and the expression (6.1.1) for the orbifold fundamental group $\pi_1^{\mathrm{orb}}(M/T,\, p_0)$.

We showed in the proof of Theorem 3.4.3 that the kernel of the monodromy homomorphism μ acts freely on $\widetilde{M/T}$. By Lemma 7.3.3 this implies that for each k the order of $\mu(\gamma_k)$ equals o_k.

Moreover, μ satisfies that $\prod \mu(\gamma_k) = 1$ since $\prod \gamma_k = 1$ and T is abelian.

The following observation is well-known.

Lemma 7.3.5. *Let W be the group of words on α_i, α_i^{-1}, β_i, β_i^{-1}, γ_j, γ_j^{-1} with $i = 1, \ldots, i_0$, $j = 1, \ldots, j_0$ for some integers i_0, j_0.*

Let G be an arbitrary group, and let $\overline{\alpha_j}$, $\overline{\beta_j}$, $\overline{\gamma_i} \in G$. Let $\Gamma \subset W$ be any subgroup and let $h_\Gamma \colon W \to \Gamma$ be the canonical homomorphism. Let $h \colon W \to G$ be the unique homomorphism such that $h(\alpha_i) = \overline{\alpha_i}$, $h(\beta_i) = \overline{\beta_i}$ and $h(\gamma_j) = \overline{\gamma_j}$. Then there exists a homomorphism $\widetilde{h} \colon \Gamma \to G$ such that $\widetilde{h} \circ h_\Gamma = h$, i.e. $\widetilde{h}$ comes from h, if and only if $\ker(h_\Gamma) \subset \ker(h)$.

Proposition 7.3.6. *Let T be a $(2n-2)$-dimensional torus. Then given a list of ingredients for T, as in Definition 7.3.1, there exists a $2n$-dimensional symplectic manifold $(M,\, \sigma)$ with an effective symplectic action of T for which at least one, and hence every T-orbit is a $(2n-2)$-dimensional symplectic submanifold of $(M,\, \sigma)$, and such that the list of ingredients of $(M,\, \sigma,\, T)$ is equal to the list of ingredients for T.*

Proof. Let $\mathcal{I}$ be a list of ingredients for the torus T, as in Definition 7.3.1. Let the pair $(\Sigma,\, \sigma^\Sigma)$ be a compact, connected symplectic orbisurface of Fuchsian signature $(g;\, \vec{o})$ given by ingredient i) of $\mathcal{I}$ in Definition 7.3.1, and with total symplectic area equal to the positive real number λ, where

λ is given by ingredient ii) of $\mathcal{I}$. By [**58**, Th. 13.3.6], since $(g;\vec{o})$ is not of the form $(0;o_1)$, $(0;o_1,o_2)$, $o_1<o_2$, Σ is a (very) good orbisurface. Let the space $\widetilde{\Sigma}$ be the orbifold universal cover of Σ, which is a smooth surface because Σ is a very good orbisurface, based at an arbitrary regular point $p_0\in\Sigma$ which we fix for the rest of the proof, c.f. the construction we gave prior to Definition 3.3.1 and Theorem 3.4.3. Let α_i, β_i, for $1\le i\le 2g$, be a symplectic basis of a free subgroup of $\mathrm{H}_1^{\mathrm{orb}}(\Sigma,\mathbb{Z})$ which together with the torsion subgroup span the entire group, and let γ_k, for $1\le k\le m$, be a geometric torsion basis and define $\mu_{\mathrm{h}}\colon \mathrm{H}_1^{\mathrm{orb}}(\Sigma,\mathbb{Z})\to T$ to be the unique homomorphism such that $\mu_{\mathrm{h}}(\alpha_i)=a_i$, $\mu_{\mathrm{h}}(\beta_i)=b_i$, $\mu_{\mathrm{h}}(\gamma_k)=c_k$, where the tuple $((a_i,b_i),(\gamma_k))$ represents ingredient iv) of $\mathcal{I}$; such a homomorphism μ_{h} exists and is well-defined because we are assuming that

$$\prod_{k=1}^{m} c_k=1,\quad c_k^{o_k}=1. \tag{7.3.2}$$

Indeed, let $h\colon W\to T$ be the homorphism on the group W of words as in Lemma 7.3.5, and let $\Gamma\subset W$ be the subgroup generated by the generators of W and the relations $\prod_{k=1}^m\gamma_k\prod_{j=1}^g[\alpha_j,\beta_j]^{-1}=1$ and $\gamma_k^{o_k}=1$ for all $k=1,\dots,m$. In the notation of Lemma 7.3.5 $\mu_{\mathrm{h}}=\widetilde{h}$. Let $h_\Gamma\colon W\to\mathrm{H}_1^{\mathrm{orb}}(\Sigma,\mathbb{Z})$ be the canonical homomorphism. Equation (7.3.2) holds if and only if $h(\prod_{k=1}^m\gamma_k^{o_k}\prod_{j=1}^g[\alpha_j,\beta_j]^{-1})=1$ and $h(\gamma_k^{o_k})=1$, for all $k=1,\dots,m$, if and only if

$$\{\prod_{k=1}^{m}\gamma_k^{o_k}\prod_{j=1}^{g}[\alpha_j,\beta_j]^{-1},\ \gamma_k^{o_k}\mid k=1,\dots,m\}\subset\ker(h)$$

if and only if $\ker(h_\Gamma)\subset\ker(h)$, and now we can apply Lemma 7.3.5 to conclude that μ_{h} is well-defined.

Let h_1 denote the orbifold Hurewicz homomorphism from $\pi_1^{\mathrm{orb}}(\Sigma,p_0)$ to $\mathrm{H}_1^{\mathrm{orb}}(\Sigma,\mathbb{Z})$. Let $\mu\colon\pi_1^{\mathrm{orb}}(\Sigma,p_0)\to T$ be the homomorphism defined as $\mu:=\mu_{\mathrm{h}}\circ h_1$. Let the orbifold fundamental group $\pi_1^{\mathrm{orb}}(\Sigma,p_0)$ act freely, see Lemma 7.3.3, on the smooth manifold given as the Cartesian product $\widetilde{\Sigma}\times T$ by the diagonal action $[\delta]\,([\gamma],t)=([\delta\,\gamma^{-1}],\mu([\delta])\cdot t)$.

Because the tuple $((a_i,b_i),(c_k))\in T^{2g+m}$ satisfies that the order of c_k is equal to o_k, we have, by Lemma 7.3.3, that this diagonal action is free and hence the bundle space defined as

$$M^{\Sigma}_{\mathrm{model}}:=\widetilde{\Sigma}\times_{\pi_1^{\mathrm{orb}}(\Sigma,p_0)}T \tag{7.3.3}$$

is a smooth manifold. The symplectic form and torus action on $M^{\Sigma}_{\mathrm{model}}$ are constructed in the exact same way as in the free case, c.f. proof of Proposition 5.3.2.

The proof that ingredients 1)–3) of $(M^{\Sigma}_{\mathrm{model}},\sigma^{\Sigma}_{\mathrm{model}})$ are equal to ingredients 1)–3) of the list $\mathcal{I}$ is the same as in the free case (c.f. proof of Proposition 5.3.2), with the observation that in the non-free case we use the classification theorem of compact, connected, smooth orientable orbisurfaces

Theorem 9.5.2 of Thurston's instead of the classical classification theorem of compact, connected, smooth surfaces, to prove that the corresponding ingredients 1) agree. We have left to show that ingredient 4) of $(M^{\Sigma}_{\text{model}}, \sigma^{\Sigma}_{\text{model}})$ equals ingredient 4) of the list $\mathcal{I}$. Let $\Omega^{\Sigma}_{\text{model}}$ stand for the flat connection on $M^{\Sigma}_{\text{model}}$ given by the symplectic orthogonal complements to the tangent spaces to the T-orbits, see Proposition 2.4.1, and let $\mu_{\text{h}}^{\Omega^{\Sigma}_{\text{model}}}$ stand for the induced homomorphism $\mu_{\text{h}}^{\Omega^{\Sigma}_{\text{model}}} \colon \mathrm{H}_1^{\text{orb}}(M^{\Sigma}_{\text{model}}/T, \mathbb{Z}) \to T$ in homology, by the monodromy of such connection. If $f \colon \mathrm{H}_1^{\text{orb}}(M^{\Sigma}_{\text{model}}/T, \mathbb{Z}) \to \mathrm{H}_1^{\text{orb}}(\Sigma, \mathbb{Z})$ is the group isomorphism induced by the orbifold symplectomorphism

$$M^{\Sigma}_{\text{model}}/T \to \widetilde{\Sigma}/\pi_1^{\text{orb}}(\Sigma, p_0) \to \Sigma, \tag{7.3.4}$$

where each arrow in (7.3.4) represents the natural map,

$$\mu_{\text{h}}^{\Omega^{\Sigma}_{\text{model}}} = \mu_{\text{h}} \circ f. \tag{7.3.5}$$

Because f is induced by a diffeomorphism, by Theorem 6.4.2 f is symplectic and torsion geometric, c.f. Definition 6.3.2 and Definition 6.3.1. Therefore there exists a unique collection of elements α_i', β_i', $1 \leq i \leq g$, in $\mathrm{H}_1^{\text{orb}}(M^{\Sigma}_{\text{model}}/T, \mathbb{Z})$ such that $f(\alpha_i') = \alpha_i$ and $f(\beta_i') = \beta_i$, for all $1 \leq i \leq g$. The elements α_i', β_i', $1 \leq i \leq g$, form a symplectic basis of a free subgroup $F^{\Omega^{\Sigma}}$ of the orbifold homology group $\mathrm{H}_1^{\text{orb}}(M^{\Sigma}_{\text{model}}/T, \mathbb{Z})$, which together with the torsion subgroup spans the entire group. Similarly let the collection γ_k', for $1 \leq k \leq m$, be such that $f(\gamma_k') = \gamma_k$, for all $1 \leq k \leq m$. The γ_k', $1 \leq k \leq m$, form a geometric torsion basis, c.f. Definition 6.1.1, such that $o_k = o'_{\tau(k)}$ for all k, $1 \leq k \leq m$, for a permutation $\tau \in \mathrm{S}_m^{\vec{o}}$. Let $\widehat{\xi}$ be the $(2g+m)$-tuple of elements $\mu_{\text{h}}^{\Omega^{\Sigma}_{\text{model}}}(\alpha_i')$, $\mu_{\text{h}}^{\Omega^{\Sigma}_{\text{model}}}(\beta_i')$, $\mu_{\text{h}}^{\Omega^{\Sigma}_{\text{model}}}(\gamma_k')$, where $1 \leq i \leq g$ and $1 \leq k \leq m$. Then by (7.3.5)

$$\widehat{\xi} = ((\mu_{\text{h}}(\alpha_i), \, \mu_{\text{h}}(\beta_i))_{i=1}^{g}, (\mu_{\text{h}}(\gamma_k))_{k=1}^{m}),$$

which in particular implies that ingredient 4) of $(M^{\Sigma}_{\text{model}}, \sigma^{\Sigma}_{\text{model}})$ is equal to $((a_i, b_i), c_k)$. $\square$

Remark 7.3.7. It is a combinatorial problem to find out, for a fixed choice of $(g; \vec{o})$, the set $T^{2g+m}_{(g;\,\vec{o})}$. For a fixed $(g; \vec{o})$, this set $T^{2g+m}_{(g;\,\vec{o})}$ may be empty: for example $T^{2+m}_{(g;\,\vec{o}=(o_1))}$ is empty. In other words, a priori the list of ingredients i)–iv) in Definition 7.3.1 may turn out to be empty for certain choices of ingredient i).

To determine which tuples $(g; \vec{o})$ give rise to a non-empty $T^{2g+m}_{(g;\,\vec{o})}$, or equivalently to a non-empty quotient $T^{2g+m}_{(g;\,\vec{o})}/\mathcal{G}_{(g,\,\vec{o})}$, is equivalent to determining which compact connected orbisurfaces can be realized as the orbit space of a symplectic T-action on a $2n$-dimensional manifold, with symplectic orbits, where T is $(2n-2)$-dimensional.

7.4. Classification theorem

By putting together the results of the previous sections, we obtain the main result of the chapter:

THEOREM 7.4.1. *Let T be a $(2n-2)$-dimensional torus. Let (M, σ) be a compact connected $2n$-dimensional symplectic manifold on which T acts effectively and symplectically and such that at least one, and hence every T-orbit is a $(2n-2)$-dimensional symplectic submanifold of (M, σ).*

Then the list of ingredients of (M, σ, T) as in Definition 7.2.1 is a complete set of invariants of (M, σ, T), in the sense that, if (M', σ') is a compact connected $2n$-dimensional symplectic manifold equipped with an effective symplectic action of T for which at least one, and hence every T-orbit is a $(2n-2)$-dimensional symplectic submanifold of (M', σ'), (M', σ') is T-equivariantly symplectomorphic to (M, σ) if and only if the list of ingredients of (M', σ', T) is equal to the list of ingredients of (M, σ, T).

And given a list of ingredients for T, as in Definition 7.3.1, there exists a symplectic $2n$-dimensional manifold (M, σ) with an effective symplectic torus action of T for which at least one, and hence every T-orbit is a $(2n-2)$-dimensional symplectic submanifold of (M, σ), such that the list of ingredients of (M, σ, T) is equal to the list of ingredients for T.

PROOF. It follows by putting together Lemma 7.2.3, Proposition 7.2.4 and Proposition 7.3.6. The combination of Lemma 7.2.3, Proposition 7.2.4 gives the uniqueness part of the theorem, while Proposition 7.3.6 gives the existence part. □

REMARK 7.4.2. The author is grateful to P. Deligne for pointing out an imprecision in a earlier version of the following statements. Let T be a $(2n-2)$-dimensional torus. Let $\mathcal{M}$ denote the category of which the objects are the compact connected symplectic $2n$-dimensional manifolds (M, σ) together with an effective symplectic T-action on (M, σ) such that at least one, and hence every T-orbit is a $(2n-2)$-dimensional symplectic submanifold of M, and of which the morphisms are the T-equivariant symplectomorphisms of (M, σ). Let $\mathcal{I}$ denote the set of all lists of ingredients as in Definition 7.3.1, viewed as a category, and of which the identities are the only endomorphisms of categories. Then the assignment ι in Definition 7.2.1 is a full functor categories from the category $\mathcal{M}$ onto the category $\mathcal{I}$. In particular the proper class $\mathcal{M}/\sim$ of isomorphism classes in $\mathcal{M}$ is a set, and the functor $\iota\colon \mathcal{M} \to \mathcal{I}$ in Definition 7.2.1 induces a bijective mapping $\iota/\sim$ from $\mathcal{M}/\sim$ onto $\mathcal{I}$. The fact that the mapping $\iota\colon \mathcal{M} \to \mathcal{I}$ is a functor and the mapping $\iota/\sim$ is injective follows from the uniqueness part of the statement of Theorem 7.4.1. The surjectivity of ι, follows from the existence part of the statement of Theorem 7.4.1.

CHAPTER 8

The four-dimensional classification

We give a classification of effective symplectic actions of 2-tori on compact connected 4-dimensional symplectic manifolds, up to equivariant symplectomorphisms, under no additional assumption.

8.1. Two families of examples

We give two families of examples of symplectic 4-manifolds.

EXAMPLE 8.1.1 (*Principal torus bundle over a torus with Lagrangian fibers).* Let T be a 2-dimensional torus. Let $T_{\mathbb{Z}}$ be the kernel of the exponential mapping $\exp : \mathfrak{t} \to T$.

a) For any choice of
   i) a discrete cocompact subgroup P of $\mathfrak{t}^*$, and
   ii) a non-zero antisymmetric bilinear mapping $c\colon \mathfrak{t}^* \times \mathfrak{t}^* \to \mathfrak{t}$ such that $c(P \times P) \subset T_{\mathbb{Z}}$,

   let $\iota\colon P \to T \times \mathfrak{t}^*$ be given by $\zeta = \zeta_1\epsilon_1 + \zeta_2\epsilon_2 \mapsto (\mathrm{e}^{-1/2\,\zeta_1\zeta_2\,c(\epsilon_1,\,\epsilon_2)},\,\zeta)$, where $\epsilon_1,\,\epsilon_2$ is a $\mathbb{Z}$-basis of P. The mapping ι is a homomorphism onto a discrete cocompact subgroup of $T \times \mathfrak{t}^*$ with respect to the non-standard standard group structure given by
   $$(t,\,\zeta)\,(t',\,\zeta') = (t\,t'\;\mathrm{e}^{-c(\zeta,\,\zeta')/2},\,\zeta + \zeta').$$
   Equip $T \times \mathfrak{t}^*$ with the standard cotangent bundle symplectic form. Then $(T \times \mathfrak{t}^*)/\iota(P)$ equipped with the action of T which comes from the action of T by translations on the left factor of $T \times \mathfrak{t}^*$, and where the symplectic form on $(T \times \mathfrak{t}^*)/\iota(P)$ is the T-invariant form induced by the symplectic form on $T \times \mathfrak{t}^*$, is a compact, connected symplectic 4-manifold on which T acts freely and for which the T-orbits are Lagrangian 2-tori.

b) For any choice of
   i) a discrete cocompact subgroup P of $\mathfrak{t}^*$, and
   ii) a homomorphism $\tau\colon P \to T$, $\zeta \mapsto \tau_\zeta$,

   let $\iota\colon P \to T \times \mathfrak{t}^*$ be given by $\zeta \mapsto (\tau_\zeta^{-1},\,\zeta)$. The mapping ι is a homomorphism onto a discrete cocompact subgroup of $T \times \mathfrak{t}^*$ with respect to the standard group structure. Equip $T \times \mathfrak{t}^*$ with the standard cotangent bundle symplectic form. Then $(T \times \mathfrak{t}^*)/\iota(P)$ equipped with the action of T which comes from the action of T by translations on the left factor of $T \times \mathfrak{t}^*$, and where the symplectic form on $(T \times \mathfrak{t}^*)/\iota(P)$ is the T-invariant form induced by the symplectic form on $T \times \mathfrak{t}^*$, is a compact, connected symplectic 4-manifold on which T acts freely with T-orbits Lagrangian 2-tori.

Indeed, it follows from the definitions that $\iota : \zeta \mapsto ({\tau_\zeta}^{-1},\, \zeta)$ in both items above is a homomorphism from P onto a discrete cocompact subgroup of $T \times \mathfrak{t}^*$. Proving that the spaces defined above are compact, connected symplectic 4-manifolds equipped with an effective symplectic action is an exercise using the definitions. Similarly, it follows from the pointwise expression for the symplectic form on $T \times \mathfrak{t}^*$ that the symplectic form on $(T \times \mathfrak{t}^*)/\iota(P)$ vanishes along the T-orbits, which hence are isotropic submanifolds of $(T \times \mathfrak{t}^*)/\iota(P)$. Let $f_P \colon T \times \mathfrak{t}^* \to (T \times \mathfrak{t}^*)/\iota(P)$ be the canonical projection map. The action on $T \times \mathfrak{t}^*$ is free, and passes to a free action on $(T \times \mathfrak{t}^*)/\iota(P)$, and hence all the T-orbits $f_P(T \times \{\xi\})$, $\xi \in \mathfrak{t}^*$, are 2-dimensional Lagrangian submanifolds of $(T \times \mathfrak{t}^*)/\iota(P)$, diffeomorphic to T.

In both cases above the projection mapping $(T \times \mathfrak{t}^*)/\iota(P) \to \mathfrak{t}^*/P$ is a principal T-bundle over the torus $\mathfrak{t}^*/P$ with Lagrangian fibers (the T-orbits). Because item ii.a) is non-trivial, the principal T-bundle is non-trivial in case a), unlike in case b). Because in this paper we are concerned with a classification up to equivariant symplectomorphisms, case b) still contains multiple non-equivalent possibilities.

Example 8.1.2 (*Principal torus orbibundle over an orbisurface with symplectic fibers*). Let T be a 2-dimensional torus. For any choice of an $(1+m)$-tuple $(g;\, \vec{o})$ of integers, where $m \geq 0$ and each component o_k of $\vec{o}$ is strictly positive, a positive real number $\lambda > 0$, a non-degenerate antisymmetric bilinear form $\sigma^{\mathfrak{t}}$ on $\mathfrak{t}$, and an element $\xi = ((a_i,\, b_i)_{i=1}^g,\, (c_k)_{k=1}^m) \in T^{2g+m}$ such that $\prod_{k=1}^m c_k = 1$ and the order of c_k is equal to o_k, let Σ be an orbisurface with Fuchsian signature $(g;\, \vec{o})$, and total symplectic area λ, and let $p_0 \in \Sigma$. The conditions on the c_k imply that Σ is a very good orbisurface, and hence $\widetilde{\Sigma}$ is a smooth surface, see Remark 7.3.2. Let α_i, β_i, for $1 \leq i \leq 2g$, be a symplectic basis of a free subgroup of $\mathrm{H}_1^{\mathrm{orb}}(\Sigma,\, \mathbb{Z})$ which together with the torsion subgroup spans the orbifold homology group $\mathrm{H}_1^{\mathrm{orb}}(\Sigma,\, \mathbb{Z})$, and let γ_k, for $1 \leq k \leq m$, be a geometric torsion basis (c.f. Definition 6.1.1). Let h_1 be the Hurewicz homomorphism. Let f_{h} be the unique homomorphism such that $f_{\mathrm{h}}(\alpha_i) = a_i$, $f_{\mathrm{h}}(\beta_i) = b_i$, $f_{\mathrm{h}}(\gamma_k) = c_k$. Let $f := f_{\mathrm{h}} \circ h_1$. Let $\pi_1^{\mathrm{orb}}(\Sigma,\, p_0)$ act freely on $\widetilde{\Sigma} \times T$ by $[\delta]\,([\gamma],\, t) = ([\delta\, \gamma^{-1}],\, f([\delta]) \cdot t)$, (see Lemma 7.3.3 for the proof of freeness). Equip the universal cover $\widetilde{\Sigma}$ with the symplectic form pullback from Σ, and $\widetilde{\Sigma} \times T$ with the product symplectic form. Let T act by translations on the right factor of $\widetilde{\Sigma} \times T$. Then the space $\widetilde{\Sigma} \times_{\pi_1^{\mathrm{orb}}(\Sigma,\, p_0)} T$ endowed with the unique symplectic form and T-action induced by the product ones is a compact, connected symplectic 4-manifold on which T acts effectively, and locally freely, and for which the T-orbits are symplectic 2-tori. The projection mapping $\widetilde{\Sigma} \times_{\pi_1^{\mathrm{orb}}(\Sigma,\, p_0)} T \to \Sigma$ is a principal T-orbibundle over the oriented orbisurface Σ.

The fact that the space we have constructed is a symplectic manifold equipped with an effective action, and that all the elements involved in the definition are well defined was checked in the proof of Proposition 7.3.6 and the references therein given. The fact that the T-orbits are 2-tori follows by the same reasoning as in Example 8.1.1.

8.2. Classification statement

The following is our main theorem: a classification, up to equivariant symplectomorphisms, of symplectic actions of 2-tori on 4-manifolds.

THEOREM 8.2.1. *Let* (M, σ) *be a compact connected symplectic* 4*-dimen-sional manifold equipped with an effective symplectic action of a* 2*-torus* T *with Lie algebra* $\mathfrak{t}$. *Then one and only one of the following cases occurs:*

1) (M, σ) *is a* 4*-dimensional symplectic toric manifold, hence determined up to* T*-equivariant symplectomorphisms by its Delzant polygon* $\mu(M)$ *centered at the origin, where* $\mu \colon M \to \mathfrak{t}^*$ *is the momentum map for the* T*-action.*
2) (M, σ) *is equivariantly symplectomorphic to a product* $\mathbb{T}^2 \times S^2$, *where* $\mathbb{T}^2 = (\mathbb{R}/\mathbb{Z})^2$ *and the first factor of* $\mathbb{T}^2$ *acts on the left factor by translations on one component, and the second factor acts on* S^2 *by rotations about the vertical axis of* S^2. *The symplectic form is a positive linear combination of the standard translation invariant form on* $\mathbb{T}^2$ *and the standard rotation invariant form on* S^2.
3) (M, σ) *is* T*-equivariantly symplectomorphic to one of the symplectic* T*-manifolds in Example 8.1.1, part a). Moreover, two such are* T*-equivariantly symplectomorphic if and only if the corresponding cocompact groups* P *and the corresponding antisymmetric bilinear forms* c *are equal.*
4) (M, σ) *is* T*-equivariantly symplectomorphic to one of the symplectic* T*-manifolds in Example 8.1.1, part b). Moreover, two such are* T*-equivariantly symplectomorphic if and only if the corresponding cocompact groups* P *and the corresponding equivalence classes* $\tau \cdot \exp(\mathrm{Sym}\,|_P) \in \mathcal{T}$ *are equal. Here* $\exp \colon \mathrm{Hom}(P, \mathfrak{t}) \to \mathrm{Hom}(P, T)$ *is the exponential map of the Lie group* $\mathrm{Hom}(P, T)$ *and* $\mathrm{Sym}\,|_P \subset \mathrm{Hom}(P, \mathfrak{t})$ *is the space of restrictions* $\alpha|_P$ *of linear maps* $\alpha \colon \mathfrak{t}^* \to \mathfrak{t}$, $\xi \mapsto \alpha_\xi$, *which are symmetric in the sense that for all* $\xi, \xi' \in \mathfrak{t}^*$, $\xi(\alpha_{\xi'}) - \xi'(\alpha_\xi) = 0$.
5) (M, σ) *is* T*-equivariantly symplectomorphic to one of the symplectic* T*-manifolds in Example 8.1.2. Moreover, two such are* T*-equivariantly symplectomorphic if and only if the corresponding* $(m+1)$*-tuple* $(g; \vec{o})$ *of integers, positive real number* $\lambda > 0$, *non-degenerate antisymmetric bilinear form* $\sigma^{\mathfrak{t}}$ *on* $\mathfrak{t}$, *and equivalence class* $\mathcal{G}_{(g,\vec{o})} \cdot \xi \in T^{2g+m}_{(g;\vec{o})}/\mathcal{G}_{(g,\vec{o})}$, *as in Definition 7.3.1, are equal.*

I am grateful to J.J. Duistermaat for suggesting dividing Example 8.1.1 into two subcases which in turn has made the statement of Theorem 8.2.1 more concrete.

REMARK 8.2.2. In Theorem 8.2.1 case 1), the T-action is Hamiltonian. In case 2) the T-action is not free, it has no fixed points, and it has one--dimensional stabilizers. In cases 3) and 4) the T-action is free. In case 5) the T-action is locally free.

REMARK 8.2.3. Theorem 8.2.1 generalizes the 4-dimensional case of Delzant's theorem [**10**] on the classification of symplectic toric 4-manifolds (i.e. symplectic 4-manifolds with a Hamiltonian 2-torus action) to symplectic actions which are not Hamiltonian.

REMARK 8.2.4. A new approach to case 1) in Theorem 8.2.1 may be found in the article of Duistermaat and the author [**13**]. Therein we describe the natural coordinatizations of a Delzant space (symplectic toric manifold) defined as a reduced phase space (symplectic geometry view-point) and give explicit formulas for the coordinate transformations. Then we explain the relation to the complex algebraic geometry view-point.

8.3. Proof of Theorem 8.2.1

Throughout we use chapters 5, 7 and ideas/methods in proofs of [**12**, Prop. 5.5, Lem. 7.1, Lem. 7.5]. Without reproving results, we have tried to be self-contained and given explicit formulas for the isomorphisms between M and the spaces 1)–5), which makes the presentation lengthier.

Step 1. First suppose that the 2-dimensional T-orbits are Lagrangian submanifolds of (M, σ). Let $\mathfrak{t}$ be the Lie algebra of T. *A Lagrangian list of ingredients $\mathcal{I}$ for T* consists of the following ingredients. 1) A subtorus T_{h} of T. 2) A Delzant polytope Δ in ${\mathfrak{t}_{\mathrm{h}}}^*$ with center of mass at the origin. 3) A discrete cocompact subgroup P of the additive subgroup $N := (\mathfrak{t}/\mathfrak{t}_{\mathrm{h}})^*$ of $\mathfrak{t}^*$. Write $T_{\mathbb{Z}}$ for the kernel of the exponential $\exp : \mathfrak{t} \to T$. 4) An antisymmetric bilinear mapping $c : N \times N \to \mathfrak{t}$ with the property that if $\zeta, \zeta' \in P$, then $c(\zeta, \zeta') \in T_{\mathbb{Z}}$. Finally, ingredient 5), the *holonomy invariant*, is an element $\overline{\tau}$ of the space $\mathcal{T}$ defined below by (8.3.2) as follows. Let $\mathrm{Hom}_c(P, T)$ denote the space of mappings $\tau : \zeta \mapsto \tau_\zeta : P \to T$ such that

$$\tau_{\zeta'}\,\tau_\zeta = \tau_{\zeta+\zeta'}\,\mathrm{e}^{c(\zeta',\,\zeta)/2}, \quad \zeta,\, \zeta' \in P. \tag{8.3.1}$$

If $h : \zeta \mapsto h_\zeta$ is a homomorphism from P to T, then $h \cdot \tau : \zeta \mapsto \tau_\zeta\, h_\zeta \in \mathrm{Hom}_c(P, T)$ for every $\tau \in \mathrm{Hom}_c(P, T)$, and $(h, \tau) \mapsto h \cdot \tau$ defines a free, proper, and transitive action of $\mathrm{Hom}(P, T)$ on $\mathrm{Hom}_c(P, T)$. For each $\zeta' \in N$, $\zeta \mapsto c(\zeta, \zeta')$ is a homomorphism from P to $\mathfrak{t}$, actually $\mathfrak{t}$-valued. Write $c(\cdot, N)$ for the set of all $c(\cdot, \zeta') \in \mathrm{Hom}(P, \mathfrak{t})$ such that $\zeta' \in N$. $c(\cdot, N)$ is a linear subspace of the Lie algebra $\mathrm{Hom}(P, \mathfrak{t})$ of $\mathrm{Hom}(P, T)$. Let Sym denote the space of all linear mappings $\alpha : \mathfrak{t}^* \to \mathfrak{t}$, $\xi \mapsto \alpha_\xi$, which are symmetric in the sense of $\xi(\alpha_{\xi'}) - \xi'(\alpha_\xi) = 0$. For each $\alpha \in \mathrm{Sym}$, the restriction $\alpha|_P$ of α to P is a homomorphism from P to $\mathfrak{t}$. In this way the set $\mathrm{Sym}\,|_P$ of all $\alpha|_P$ such that $\alpha \in \mathrm{Sym}$ is another linear subspace of $\mathrm{Hom}(P,\,\mathfrak{t})$. Write

$$\mathcal{T} := \mathrm{Hom}_c(P,\,T)/\exp\mathcal{A}, \quad \mathcal{A} := c(\cdot,\,N) + \mathrm{Sym}\,|_P \tag{8.3.2}$$

for the orbit space of the action of the Lie subgroup $\exp\mathcal{A}$ of $\mathrm{Hom}(P,\,T)$ on $\mathrm{Hom}_c(P,\,T)$. The following is a consequence of [**12**, Thms. 9.4, 9.6] (it is the case $\mathfrak{l} = \mathfrak{t}$ therein).

Proposition 8.3.1. *Every list of ingredients $\mathcal{I}$ as above gives rise to a compact connected symplectic 4-manifold on which T acts symplectically and T_{h} acts Hamiltonianly. If T_{f} is a complementary subtorus to T_{h} in T, T_{f} acts freely on this manifold. Additionally, the 2-dimensional T-orbits are Lagrangian submanifolds. Moreover, different lists $\mathcal{I}$ of ingredients give rise to non T-equivariantly symplectomorphic symplectic manifolds.*

Following [**12**] we construct a symplectic manifold equipped with a torus action as in Proposition 8.3.1. First we define a smooth manifold, then a symplectic form on it, and finally we equip it with a torus action. Let

$$c\colon N \times N \to \mathfrak{t} \tag{8.3.3}$$

be an antisymmetric bilinear mapping as in ingredient 4) of $\mathcal{I}$. Then $\mathfrak{g} := \mathfrak{t} \times N$ equipped with the operation

$$[(X,\,\zeta),\,(X',\,\zeta')] = -(c(\zeta,\,\zeta'),\,0), \quad (X,\,\zeta),\,(X',\,\zeta') \in \mathfrak{g} = \mathfrak{t} \times N,$$

is a 2-step nilpotent Lie algebra, and $(t,\,\zeta)\,(t',\,\zeta') = (t\,t'\,\mathrm{e}^{-c(\zeta,\,\zeta')/2},\,\zeta + \zeta')$ defines a product in

$$G := T \times N \tag{8.3.4}$$

for which G is a Lie group with Lie algebra $\mathfrak{g}$. Choose an element $\tau \in \mathrm{Hom}_c(P,\,T)$ such that $\overline{\tau} = (\exp\mathcal{A}) \cdot \tau$, see (8.3.2). Because the τ_ζ, $\zeta \in P$, satisfy (8.3.1), it follows that

$$H := \{(t,\,\zeta) \in G \mid \zeta \in P \text{ and } t\,\tau_\zeta \in T_{\mathrm{h}}\} \tag{8.3.5}$$

is a closed Lie subgroup of G and that

$$((t,\,\zeta),\,x) \mapsto (t\,\tau_\zeta) \cdot x : H \times M_{\mathrm{h}} \to M_{\mathrm{h}} \tag{8.3.6}$$

defines a smooth action of H on the Delzant manifold $(M_{\mathrm{h}},\,\sigma_{\mathrm{h}},\,T_{\mathrm{h}})$ associated to the polytope $\Delta \subset (\mathfrak{t}_{\mathrm{h}})^*$ by Delzant's theorem [**10**]. The right action of H on G is proper and free because H is a closed Lie subgroup of G, and hence the action of H on $G \times M_{\mathrm{h}}$ defined by $h\,(g,\,x) = (g\,h^{-1},\,h\cdot x)$ is proper and free. The quotient

$$M_{\mathrm{model}} := G \times_H M_{\mathrm{h}} \tag{8.3.7}$$

has a unique structure of a smooth manifold for which the canonical projection $\pi : G \times M_{\mathrm{h}} \to G \times_H M_{\mathrm{h}}$ is a principal H-bundle. Since $G \times M_{\mathrm{h}}$ is connected and π is continuous, $G \times_H M_{\mathrm{h}}$ is connected. The projection $(g,\,x) \mapsto g$ induces a G-equivariant smooth fibration $\psi : G \times_H M_{\mathrm{h}} \to G/H$

with fiber M_{h}, the fiber bundle induced from the principal fiber bundle $G \to G/H$ by means of the action of H on M_{h}. Because P is cocompact in N, G/H is compact, and since the fiber M_{h} is compact, $G \times_H M_{\mathrm{h}}$ is compact.

We now define the symplectic form on $G \times_H M_{\mathrm{h}}$. Let T_{f} be any complementary subtorus to T_{h} in T, and let $\mathfrak{t}_{\mathrm{f}}$ be its Lie algebra. Let $\mu\colon M_{\mathrm{h}} \to \Delta$ be the momentum map of the Hamiltonian T_{h}-action. Let X_{h} denote the $\mathfrak{t}_{\mathrm{h}}$-component of X in the decomposition $\mathfrak{t}_{\mathrm{h}} \oplus \mathfrak{t}_{\mathrm{f}}$. Let c_{h} denote the $\mathfrak{t}_{\mathrm{h}}$-component of c in $\mathfrak{t} = \mathfrak{t}_{\mathrm{h}} \oplus \mathfrak{t}_{\mathrm{f}}$. Write $\delta a = ((\delta t,\, \delta\zeta),\, \delta x)$ and $\delta' a = ((\delta' t,\, \delta'\zeta),\, \delta' x)$ for two tangent vectors to $G \times M_{\mathrm{h}}$ at $a = ((t,\, \zeta),\, x)$, where we identify each tangent space of the torus T with $\mathfrak{t}$. Write $X = \delta t + c(\delta\zeta,\, \zeta)/2$ and $X' = \delta' t + c(\delta'\zeta,\, \zeta)/2$. Define

$$\begin{aligned} \omega_a(\delta a,\, \delta' a) \;&=\; \delta\zeta(X') - \delta'\zeta(X) \\ &\quad -\mu(x)(c_{\mathrm{h}}(\delta\zeta,\, \delta'\zeta)) + (\sigma_{\mathrm{h}})_x(\delta x,\, ({X'}_{\mathrm{h}})_{M_{\mathrm{h}}}(x)) \\ &\quad -(\sigma_{\mathrm{h}})_x(\delta' x,\, (X_{\mathrm{h}})_{M_{\mathrm{h}}}(x)) + (\sigma_{\mathrm{h}})_x(\delta x,\, \delta' x). \end{aligned} \tag{8.3.8}$$

It follows from [**12**, Proof of Thm. 9.6] that ω is a basic 2-form for the action of H on $G \times M_{\mathrm{h}}$ and it descends to a symplectic form σ_{model} on $G \times_{\mathrm{h}} M_{\mathrm{h}}$.

Finally, the definition of the T-action on $G \times_H M_{\mathrm{h}}$ is as follows. On $G \times M_{\mathrm{h}}$ we have the action of $s \in T$ which sends $((t,\, \zeta),\, x)$ to $((s\,t,\, \zeta),\, x)$. The induced action of T on $G \times_H M_{\mathrm{h}}$ leaves σ_{model} invariant. The torus T_{h} acts on $G \times_H M_{\mathrm{h}}$ in a Hamiltonian fashion, c.f. [**12**, Proof of Thm. 9.6], and the complementary subtorus T_{f} to T_{h} in T acts freely.

Step 2. Following [**12**, Thm. 9.4] we sketch a proof of the following. We will use the same proof method in Case 3.2 in Step 3, and hence why it is appropriate to exhibit this proof here.

PROPOSITION 8.3.2. *Let* $(M,\, \sigma)$ *be a compact connected symplectic* 4-*-manifold, the* 2*-dimensional orbits of which are Lagrangian submanifolds. Then there exists a* T*-equivariant symplectomorphism from* $(M_{\mathrm{model}} := G \times_H M_{\mathrm{h}}, \sigma_{\mathrm{model}})$ *to* $(M,\, \sigma)$*, for a unique choice of a list of ingredients for* T *as above.*

Sketch of Proof. The first observation is that the orbit space M/T is a polyhedral $\mathfrak{t}^*$-parallel space. A $\mathfrak{t}^*$*-parallel space* is a Hausdorff topological space modelled on a corner of $\mathfrak{t}^*$, c.f. [**12**, Def. 10.1]. The local charts ϕ_α into $\mathfrak{t}^*$, satisfy that the mapping $x \mapsto \phi_\alpha(x) - \phi_\beta(x)$ is locally constant for all values of α and β. For $X \in \mathfrak{t}$, consider the form $\hat{\sigma}(X) := -\mathrm{i}_{X_M}\sigma \in \Omega^1(M)$, which is a closed, basic form. Write M_{reg} for the subset of M where the T-action is free. The assignment $\hat{\sigma}\colon x \mapsto (\hat{\sigma}_x\colon \mathrm{T}_x\, M \to \mathfrak{t}^*)$, where we use the identification $\hat{\sigma}_x \simeq \mathrm{T}_x\, \pi$, induces an isomorphism $\hat{\sigma}_p\colon\ \mathrm{T}_p(M_{\mathrm{reg}}/T) \to \mathfrak{t}^*$. This implies that a constant vector field on $\mathcal{X}^\infty(M_{\mathrm{reg}}/T)$ may be thought of as an element $\xi \in \mathfrak{t}^*$. This parallel structure gives a natural action $+$ of $\mathfrak{t}^*$ on M/T, which we write $p \mapsto p + \xi$, by traveling from p for time 1 in the direction of ξ; this action is only well-defined on the subspace $N = (\mathfrak{t}/\mathfrak{t}_{\mathrm{h}})^*$ of $\mathfrak{t}^*$, i.e. on those vectors that do not point in the Hamiltonian direction $\mathfrak{t}^*_{\mathrm{h}} \subset \mathfrak{t}^*$, as otherwise we hit the boundary of M/T. We call P the period lattice of this N-action on M/T. The quotient N/P is a torus, and we have used the same letters N and P as in the previous abstract list of ingredients because they play this exact role. P is rigorously introduced in [**12**, Lem. 10.12, Prop. 3.8]. In this way $L_\xi \in \mathcal{X}^\infty(M_{\mathrm{reg}})$ is a *lift of* ξ if $\hat{\sigma}_x(L_\xi) = \xi$. Secondly, as $\mathfrak{t}^*$-parallel spaces, there is an isomorphism $M/T \simeq \Delta \times S$, where Δ is a Delzant polytope, and S is a torus, c.f. [**12**, Prop. 3.8, Th. 10.12]. Concretely S is the torus N/P, and $\Delta \subset \mathfrak{t}^*_{\mathrm{h}}$ is the Delzant polytope associated to the maximal Hamiltonian torus action on M, the action of $T_{\mathrm{h}} \subset T$. The description and classification of this parallel structure involves the classification of V-parallel spaces. Moreover, it involves generalizing the Tietze-Nakajima theorem in [**55**], [**42**].

In [**12**, Prop. 5.5] we showed that there exists a nice and so called admissible connection $\xi \in \mathfrak{t}^* \mapsto L_\xi \in \mathcal{X}^\infty(M_{\mathrm{reg}})$, [**12**, Def. 5.3], for the principal T-bundle $\pi\colon M_{\mathrm{reg}} \to M_{\mathrm{reg}}/T$. Here the lifts L_ξ, $\xi \in N$, have smooth extensions to M and by "nice" we mean that the connection has simple Lie brackets $[L_\xi,\, L_\eta]$ associated to it. More precisely, there exists a unique antisymmetric bilinear form $c\colon N \times N \to \mathfrak{t}$ such that $[L_\xi,\, L_\eta] = c(\xi,\, \eta)_M$, if $\xi,\, \eta \in N$, where c corresponds to (8.3.3) and represents the Chern class of $\pi\colon M_{\mathrm{reg}} \to M_{\mathrm{reg}}/T$, and $[L_\xi,\, L_\eta] = 0$ if $\xi,\, \eta \in \mathfrak{t}^*$, $\xi,\, \eta \notin N$. We also require this nice connection to have simple symplectic pairings $\sigma(L_\xi,\, L_\eta)$: in the particular case that $c_{\mathrm{h}} = 0$, which is the one we shall need later, the condition is that $\sigma(L_\xi,\, L_\eta) = 0$, for all $\xi,\, \eta \in \mathfrak{t}^*$. The lifts L_ξ, $\xi \notin N$, are singular on $M \setminus M_{\mathrm{reg}}$, and the singularities are required to be simple. From this antisymmetric bilinear form c and the space N in the previous paragraph, we construct the group G in Step 1, and equip it with the non-standard operation (8.3.4).

Then in [**12**, Prop. 6.1] we define the integrable distribution$\{D_x\}_{x \in M}$on M where D_x is the span of $L_\eta(x)$, $Y_M(x)$ as $Y \in \mathfrak{t}_{\mathrm{h}}$, $\eta \in C$, where C is a complementary subspace to N in $\mathfrak{t}^*$. The integral manifolds of this distribution are all compact[1] connected symplectic manifolds (with the restricted symplectic form) on which T_{h} acts Hamiltonianly, and we fix one of them which we call $(M_{\mathrm{h}},\, \sigma_{\mathrm{h}},\, T_{\mathrm{h}})$.

Assuming this, we make the definition of H in (8.3.5), and M_{model} in (8.3.7) from the connection $\xi \mapsto L_\xi$; the definition of H involves the holonomy of the connection $\zeta \mapsto L_\zeta$, which is defined as follows: for each $\zeta \in P$ and $p \in M/T$, the curve $\gamma_\zeta(t) := p + t\,\zeta$, $0 \leq t \leq 1$, is a loop in M/T. If $x \in M$ and $p = \pi(x)$, then the curve $\delta(t) = \mathrm{e}^{t\,L_\zeta}(x)$, $0 \leq t \leq 1$, is called the *horizontal lift in* M *of the loop* γ_ζ *which starts at* x, because $\delta(0) = x$ and $\delta'(t) = L_\zeta(\delta(t))$ is a horizontal tangent vector which is mapped by $\mathrm{T}_{\delta(t)}\pi$ to the constant vector ζ, which implies that $\pi(\delta(t)) = \gamma_\zeta(t)$, $0 \leq t \leq 1$. The

[1]This is remarkable because the vector fields which define the distribution do blow up at many points

element of T which maps the initial point $\delta(0) = x$ to the end point $\delta(1)$ is $\tau_\zeta(x)$. Because $\delta(1) = \mathrm{e}^{L_\zeta}(x)$, we have $\tau_\zeta(x) \cdot x = \mathrm{e}^{L_\zeta}(x)$. This defines a map $\tau\colon P \to T$, $\zeta \mapsto \tau_\zeta(x) := \tau_\zeta$, which corresponds to ingredient 5) in Step 1. This map depends on x and on the connection $\xi \in \mathfrak{t}^* \mapsto L_\xi$, which is not unique, and the actual invariant is an equivalence class of such maps. Details appeared in [**12**, Prop. 7.2, Lem. 7.1, Sec. 7.5]. In Case 3.2 below we shall slightly change the connection to a more convenient one that still satisfies the properties on the Lie brackets and symplectic pairings above.

Then we define the symplectic form σ_{model} as the form descending from (8.3.8). The T-equivariant symplectomorphism between the model and the symplectic T-manifold $(M,\, \sigma)$ is induced by

$$((t,\, \xi),\, x) \mapsto t \cdot \mathrm{e}^{L_\xi}(x)\colon G \times M_{\mathrm{h}} \to M. \tag{8.3.9}$$

□

The next step combines steps 1,2 with chapters 5, 7.

Step 3. Suppose that $(M,\, \sigma)$ is a compact and connected symplectic 4--dimensional manifold equipped with an effective symplectic action of a 2-torus T. If there exists a 2-dimensional symplectic T-orbit then, by Lemma 2.2.2, every T-orbit is a 2-dimensional symplectic submanifold of $(M,\, \sigma)$. Assume that none of the T-orbits is a symplectic 2-dimensional submanifold of $(M,\, \sigma)$. Then the antisymmetric bilinear form $\sigma^{\mathfrak{t}}$ in (2.1.1) is degenerate, and hence it has a one or two-dimensional kernel $\mathfrak{l} \subset \mathfrak{t}$. If $\dim \mathfrak{l} = 2$, then $\mathfrak{l} = \mathfrak{t}$ and $\sigma^{\mathfrak{t}} = 0$, so every T-orbit is an isotropic submanifold of $(M,\, \sigma)$. Hence the 2-dimensional T-orbits are Lagrangian submanifolds of $(M,\, \sigma)$. If $\dim \mathfrak{l} = 1$ there exists a one-dimensional complement V to $\mathfrak{l}$ in $\mathfrak{t}$, such that the restriction of $\sigma^{\mathfrak{t}}$ to V is a non-degenerate, antisymmetric bilinear form, and hence identically equal to zero, a contradiction. Hence either every T-orbit is a 2-dimensional symplectic submanifold of $(M,\, \sigma)$, or the 2-dimensional T-orbits are Lagrangian submanifolds of $(M,\, \sigma)$. We distinguish four cases according to this.

Case 3.1. Suppose that the action of T on M is Hamiltonian. Because T is 2-dimensional, $(M,\, \sigma)$ is a symplectic toric manifold, and hence by Delzant's theorem [**10**], the image $\mu(M)$ of M under the momentum map $\mu\colon M \to \mathfrak{t}^*$ determines $(M,\, \sigma)$ up to T-equivariant symplectomorphisms (the explicit construction of M from $\mu(M)$ is given in Delzant's article).

In the next three cases we assume that the T-action on M is not Hamiltonian, so that $(M,\, \sigma)$ is not a symplectic toric manifold.

Case 3.2. Suppose that $(M,\, \sigma)$ has Lagrangian 2-dimensional orbits, and that T does not act freely on $(M,\, \sigma)$. Since the T action is not free, the Hamiltonian torus T_{h} introduced in Step 1 is a 1-dimensional subtorus of T.

Let T_{f} be a complementary torus to the Hamiltonian torus T_{h} in T, and let $\mathfrak{t}_{\mathrm{f}}$ be its Lie algebra. Since $\mathfrak{t}$ is 2-dimensional and $N = (\mathfrak{t}/\mathfrak{t}_{\mathrm{h}})^*$, $\dim N = 1$. Therefore the antisymmetric bilinear form $c\colon N \times N \to \mathfrak{t}$ in (8.3.3) is identically zero. Then the mapping $\iota : \zeta \mapsto ({\tau_\zeta}^{-1}, \zeta)$ is a homomorphism from P onto a discrete cocompact subgroup of G_{f}. Write $M_{\mathrm{f}} := G_{\mathrm{f}}/\iota(P)$. We have that the complement C to N in $\mathfrak{t}^*$ is of the form $C = \mathbb{R}\,\eta$ for a nonzero $\eta \in C$, and $P = \mathbb{Z}\,\epsilon$ for a nonzero $\epsilon \in P$, unique up to its sign. We write $\tau := \tau_\epsilon$. Replace L_η and L_ϵ by $L'_\eta := L_\eta + U_M$ and $L'_\epsilon := L_\epsilon + V_M$ for suitable $U,\, V \in \mathfrak{t}$. In this case $\sigma(L'_\eta,\, L'_\epsilon) = \sigma(L_\eta,\, V_M) + \sigma(U_M,\, L_\epsilon) = \eta(V) - \epsilon(U)$. We can arrange $\epsilon(U) = \eta(V)$ for any desired $V \in \mathfrak{t}$ by choosing U appropriately, because ϵ is nonzero. If we choose $V \in \mathfrak{t}$ such that $\tau \exp V = 1$, as it always can be done, then $\mathrm{e}^{L'_\epsilon}(x) = \mathrm{e}^{V_M} \circ L_\epsilon(x) = \exp(V)\cdot\tau\cdot x = (\exp(V)\,\tau)\cdot x = x$, hence $\tau = 1$. So we have shown that by going back and replacing the lifts in Step 2 by these new ones we get $\tau = \tau_\epsilon = 1$.

The Lie group G in (8.3.4) is the Cartesian product $T_{\mathrm{h}} \times G_{\mathrm{f}}$, in which $G_{\mathrm{f}} := T_{\mathrm{f}} \times N$, where the product in G_{f} is defined by $(t_{\mathrm{f}},\, \zeta)(t'_{\mathrm{f}},\, \zeta') = (t_{\mathrm{f}}\, t'_{\mathrm{f}},\, \zeta + \zeta')$. With the same proof as in [**12**, Prop. 7.2] that (8.3.9) induces a T-equivariant diffeomorphism $G \times_H M_{\mathrm{h}} \to M$, we obtain that the mapping $((t_{\mathrm{f}},\, \zeta), x) \mapsto t_{\mathrm{f}} \cdot \mathrm{e}^{L_\zeta}(x) : G_{\mathrm{f}} \times M_{\mathrm{h}} \to M$ induces a T-equivariant diffeomorphism α_{f} from $M_{\mathrm{f}} \times M_{\mathrm{h}}$ onto M. Recall that c_{h} denotes the $\mathfrak{t}_{\mathrm{h}}$-component of c in $\mathfrak{t} = \mathfrak{t}_{\mathrm{h}} \oplus \mathfrak{t}_{\mathrm{f}}$, and which since c is zero, it is zero. Let π_{f}, π_{h} be the projection from $M_{\mathrm{f}} \times M_{\mathrm{h}}$ onto the first and the second factor, respectively. The symplectic form ${\alpha_{\mathrm{f}}}^*\,\sigma$ on $M_{\mathrm{f}} \times M_{\mathrm{h}}$ is equal to ${\pi_{\mathrm{f}}}^*\,\sigma_{\mathrm{f}} + {\pi_{\mathrm{h}}}^*\,\sigma_{\mathrm{h}}$, and the symplectic form σ_{f} on M_{f} is given, according to (8.3.8) with $c_{\mathrm{h}} = 0$, by

$$(\sigma_{\mathrm{f}})_b(\delta b,\, \delta' b) = \delta\zeta(\delta' t) - \delta'\zeta(\delta t). \tag{8.3.10}$$

Here $b = (t,\, \zeta)\,\iota(P) \in G_{\mathrm{f}}/\iota(P)$ and the tangent vectors $\delta b = (\delta t,\, \delta\zeta)$ and $\delta' b = (\delta' t,\, \delta'\zeta)$ are elements of $\mathfrak{t}_{\mathrm{f}} \times N$. It follows that $(M,\, \sigma,\, T)$ is T-equivariantly symplectomorphic to $(M_{\mathrm{f}},\, \sigma_{\mathrm{f}},\, T_{\mathrm{f}}) \times (M_{\mathrm{h}},\, \sigma_{\mathrm{h}},\, T_{\mathrm{h}})$, in which $(M_{\mathrm{f}},\, \sigma_{\mathrm{f}},\, T_{\mathrm{f}})$ is a compact connected symplectic manifold with a free symplectic action T_{f}-action. Here $t \in T$ acts on $M_{\mathrm{f}} \times M_{\mathrm{h}}$ by sending $(x_{\mathrm{f}},\, x_{\mathrm{h}})$ to $(t_{\mathrm{f}} \cdot x_{\mathrm{f}},\, t_{\mathrm{h}} \cdot x_{\mathrm{h}})$, if $t = t_{\mathrm{f}}\, t_{\mathrm{h}}$ with $t_{\mathrm{f}} \in T_{\mathrm{f}}$ and $t_{\mathrm{h}} \in T_{\mathrm{h}}$.

Then the classification of symplectic toric manifolds [**10**] of Delzant implies that M_{h} is a 2-sphere equipped with an S^1-action by rotations about the vertical axis and endowed with a rotationally invariant symplectic form. On the other hand M_{f} is diffeomorphic to $T \simeq \mathbb{T}^2$, since S^1 does not act freely on a surface non-diffeomorphic to T, which in turn implies that M is of the form given in part 2) of the statement.

Case 3.3. Suppose that $(M,\, \sigma)$ *has Lagrangian* T*-orbits and that* T *acts freely on* $(M,\, \sigma)$. The claim that $(M,\, \sigma)$ is as in part i) of the statement for a unique choice of the ingredients therein, follows from Proposition 8.3.1 once we show that the model $G \times_H M_{\mathrm{h}}$ of $(M,\, \sigma)$ in Propositon 8.3.2 is of the form $(T \times \mathfrak{t}^*)/\iota(P)$ with the T-actions and symplectic form in part i), which we do next. Indeed, since the T-action is free, $T_{\mathrm{h}} = \{1\}$ and hence, by (8.3.5), $H = \{(t, \tau_\zeta) \mid \zeta \in P,\, t\,\tau_\zeta = 0\} = \iota(P)$ and $\mathfrak{t}_{\mathrm{h}} = \{0\}$, where ι was given in part i) of the statement. Also $G = T \times \mathfrak{t}^*$, c.f. (8.3.4). Because $T_{\mathrm{h}} = \{1\}$, the Delzant submanifold M_{h} may be chosen to be any point $x \in M$, the Delzant polytope associated to M_{h} equals $\{0\} \subset \mathfrak{t}_{\mathrm{h}}^*$, and there is a natural T-equivariant symplectomorphism

$$G \times_H M_{\mathrm{h}} \to G/H \times \{x\} \to G/H = (T \times \mathfrak{t}^*)/\iota(P),$$

where $(T \times \mathfrak{t}^*)/\iota(P)$ is equipped with the symplectic form (8.3.10), but considering that $\delta b = (\delta t,\, \delta\zeta)$, $\delta' b = (\delta' t,\, \delta'\zeta) \in \mathfrak{t} \times \mathfrak{t}^*$. We obtain such an expression for the symplectic form by simplifying expression (8.3.8) according to σ_{h}, c_{h} trivial. The mapping $(t,\, \zeta) \mapsto t\,\mathrm{e}^{L_\zeta}(x) \colon T \times \mathfrak{t}^* \to M$, induced by the mapping (8.3.9) induces a T-equivariant symplectomorphism between $(T \times \mathfrak{t}^*)/\iota(P)$ and $(M,\, \sigma)$.

To conclude the proof we need to show that if the antisymmetric bilinear form c is non-zero, then the space $\mathcal{T}$ consists of a single point, the class of $\tau \colon \zeta \mapsto \mathrm{e}^{1/2\,\zeta_1\zeta_2 c(\epsilon_1,\epsilon_2)}$, where ϵ_1, ϵ_2 form a $\mathbb{Z}$-basis of P. One can check that $\tau \in \mathrm{Hom}_c(P,\, T)$. It is left to show that $c(\cdot,\, \mathfrak{t}^*) + \mathrm{Sym}\,|_P = \mathrm{Hom}(P,\, \mathfrak{t})$, which would imply that $\mathcal{A} = \mathrm{Hom}(P, \mathfrak{t})$, hence $\exp \mathcal{A} = \mathrm{Hom}(P,\, T)$. Because $\mathrm{Hom}(P,\, T)$ acts transitively on $\mathrm{Hom}_c(P,\, T)$, it follows that $\mathcal{T} = \mathrm{Hom}_c(P, T)/\exp \mathcal{A}$ is trivial in the sense that it consists of a single point.

Next we check that $c(\cdot,\, \mathfrak{t}^*) + \mathrm{Sym}\,|_P = \mathrm{Hom}(P,\, \mathfrak{t})$. Indeed, let ϵ^1, ϵ^2 be a $\mathbb{Z}$-basis of P, and let ρ_1, ρ_2 be its dual basis defined by $\epsilon^i(\rho_j) = \delta^i_j$. Using coordinates in $\mathfrak{t}$ and $\mathfrak{t}^*$ with respect to these bases, $\mathrm{Hom}(P,\, \mathfrak{t})$ is identified with the 4-dimensional vector space of all 2-dimensional square matrices with real coefficients, and the subspace $\mathrm{Sym}\,|_P$ of it is the 3-dimensional linear subspace of symmetric 2-dimensional matrices. The space $c(\cdot,\, \mathfrak{t}^*)$ is the linear subspace of all homomorphisms $\zeta \mapsto c(\zeta,\, \xi)$ from P to $\mathfrak{t}$, where $\xi \in \mathfrak{t}^*$. The first and second column of the matrix of such a homomorphism are equal to $c(\epsilon_1,\, \xi) = \xi_2\, c(\epsilon_1,\, \epsilon_2)$ and $c(\epsilon_2,\, \xi) = -\xi_1\, c(\epsilon_1,\, \epsilon_2)$, where the column vector $c(\epsilon_1,\, \epsilon_2)$ has the entries c_1, c_2 such that $c(\epsilon_1,\, \epsilon_2) = c_1\rho_1 + c_2\rho_2$. If $c(\epsilon_1,\, \epsilon_2) = 0$, then c is identically zero, a contradiction. Therefore the matrix with $\xi_2 = 1$ and $\xi_1 = 0$ is not symmetric, which implies that $c(\cdot,\, \mathfrak{t}^*) + \mathrm{Sym}\,|_P = \mathrm{Hom}(P,\, \mathfrak{t})$, and therefore $\mathcal{T}$ is trivial.

Case 3.4. Suppose that all T-orbits are symplectic 2*-tori.* Then it follows from Theorem 7.4.1 that $(M,\, \sigma)$ is T-equivariantly symplectomorphic to the symplectic T-manifold given in part ii) of the statement for a unique choice of ingredients, and this unique list is given in Definition 7.2.1 with $\mu = f$ and $\mu_{\mathrm{h}} = f_{\mathrm{h}}$.

This concludes the proof of the theorem. □

Remark 8.3.3. The approaches to the proofs of Case 3.2 and Case 3.3 above are different. In Case 3.3, the Delzant manifold M_{h} is trivial, so the proof is obtained as a particular case of the model $G \times_H M_{\mathrm{h}}$. In Case 3.2 this approach does not lead to a product $M_{\mathrm{f}} \times M_{\mathrm{h}}$ directly, and hence why we used the same proof method as in Proposition 8.3.2. Although if $M_{\mathrm{f}} := G_{\mathrm{f}}/\iota(P)$ and $H^o := T_{\mathrm{h}} \times \{1\}$, the mapping $g \mapsto g\,H^o$ defines an isomorphism from G_{f} onto the group G/H^o, and an isomorphism from $\iota(P)$ onto H/H^o, which leads to an identification of M_{f} with G/H, $G \times_H M_{\mathrm{h}}$ is not even T-equivariantly diffeomorphic to $G/H \times M_{\mathrm{h}}$ with the induced action.

8.4. Corollaries of Theorem 8.2.1

In the statement of Theorem 8.2.1, $(T \times \mathfrak{t}^*)/\iota(P)$ is a principal T-bundle over the torus $\mathfrak{t}^*/P$. Palais and Stewart [**47**] showed that every principal torus bundle over a torus is diffeomorphic to a nilmanifold for a two-step nilpotent Lie group. We have given an explicit description of this nilmanifold structure in Example 8.1.1. Theorem 8.2.1 also implies the following results.

THEOREM 8.4.1. *The only compact connected 4-dimensional symplectic manifold equipped with a non-locally-free and non-Hamiltonian effective symplectic action of a 2-torus is, up to equivariant symplectomorphisms, the product $\mathbb{T}^2 \times S^2$, where $\mathbb{T}^2 = (\mathbb{R}/\mathbb{Z})^2$ and the first factor of $\mathbb{T}^2$ acts on the left factor by translations on one component, and the second factor acts on S^2 by rotations about the vertical axis of S^2. The symplectic form is a positive linear combination of the standard translation invariant form on $\mathbb{T}^2$ and the standard rotation invariant form on S^2.*

PROOF. Since the T-action is not Hamiltonian, case 1) in the statement of Theorem 8.2.1 cannot occur. Since the action is non-locally-free, there are one-dimensional or two-dimensional stabilizer subgroups, and (M, σ) cannot be as in item 3), 4) or 5): in item 3) and item 4) the stabilizers are all trivial, and in item 5) the stabilizers are finite groups. □

THEOREM 8.4.2. *Let (M, σ) be a non-simply connected, compact connected symplectic 4-manifold equipped with a symplectic non-free action of a 2-torus T and such that M is not homeomorphic to $T \times S^2$. Then (M, σ) is a principal T-orbibundle over a good orbisurface with symplectic fibers.*

Moreover, (M, σ) is of the form given in Example 8.1.2.

PROOF. This follows by Theorem 8.2.1 by the fact that Delzant manifolds are simply connected. Indeed, every Delzant manifold can be provided with the structure of a toric variety defined by a complete fan, c.f. Delzant [**10**] and Guillemin [**18**, App. 1], and Danilov [**9**, Th. 9.1] observed that such a toric variety is simply connected. The argument is that the toric variety has an open cell which is isomorphic to $\mathbb{C}^n$, of which the complement is a complex subvariety of complex codimension one. Hence any loop can be deformed into the cell and contracted within the cell to a point. □

REMARK 8.4.3. The reasons because of which we have imposed that the torus T is 2-dimensional and (M, σ) is 4-dimensional in Theorem 8.2.1 are the following.

i) There does not exist a classification of n-dimensional smooth orbifolds if $n > 2$.
ii) In dimensions greater than 2, the symplectic form is not determined by a single number (Moser's theorem).
iii) If M is not 4-dimensional, and T is not 2-dimensional, then there are many cases where not all of the torus orbits are symplectic, and not all of the torus orbits are isotropic. Other than that Theorem 8.2.1 may be generalized. Let (M, σ) be a compact connected $2n$--dimensional symplectic manifold equipped with an effective symplectic action of a torus T and suppose that one of the following two conditions hold.
   (1) There exists a T-orbit of dimension $\dim T$ which is a symplectic submanifold of (M, σ).
   (2) There exists a principal T-orbit which is a coisotropic submanifold of (M, σ).

   These symplectic manifolds with T-actions are classified analogously to Theorem 8.2.1, but in weaker terms (e.g. involving an n-dimensional orbifold as in item i) above instead of the first two items of Definition 7.3.1).

CHAPTER 9

Appendix: (sometimes symplectic) orbifolds

There does not appear to be a universally accepted definition of an orbifold, so for the sake of being precise we do not use terminology in orbifolds without clarifying or introducing it. We introduce only the concepts we explicitly use, and for the spaces we use them, so this is not intended to be a general appendix on orbifolds but rather an attempt to provide precise definitions for the terms we use in the particular symplectic setting of the paper.

9.1. Bundles, connections

9.1.1. Orbifolds and diffeomorphisms. Following largely but not entirely Boileau-Maillot-Porti [**6**, Sec. 2.1.1], we define orbifold. We also borrow from ideas in Satake [**49**], [**50**] and Thurston [**57**]. Our definition of orbifold is close to that in Haefliger's paper [**20**, Sec. 4]. Unfortunately Haefliger's paper does not appear to be so well known and is frequently not given proper credit. I thank Y. Karshon for making me aware of this.

Definition 9.1.1. *A* smooth n-dimensional orbifold $\mathcal{O}$ *is a metrizable topological space* $|\mathcal{O}|$ *endowed with an equivalence class of orbifold atlases. An* orbifold atlas *is a collection* $\{(U_i, \widetilde{U}_i, \phi_i, \Gamma_i)\}_{i\in I}$ *where for each* $i \in I$, U_i *is an open subset of* $|\mathcal{O}|$, $\widetilde{U}_i$ *is an open and connected subset of* $\mathbb{R}^n$, $\phi_i \colon \widetilde{U}_i \to U_i$ *is a continuous map, called an* orbifold chart, *and* Γ_i *is a finite group of diffeomorphisms of* $\widetilde{U}_i$, *satisfying:*

i) *the* U_i*'s cover* $|\mathcal{O}|$,
ii) *each* ϕ_i *factors through a homeomorphism between* $\widetilde{U}_i/\Gamma_i$ *and* U_i, *and*
iii) *the charts are compatible. This means that for each* $x \in \widetilde{U}_i$ *and* $y \in \widetilde{U}_j$ *with* $\phi_i(x) = \phi_j(y)$, *there is a diffeomorphism* ψ *between a neighborhood of* x *and a neighborhood of* y *such that* $\phi_j(\psi(z)) = \phi_i(z)$ *for all* z *in such a neighborhood.*

Two orbifold atlases are equivalent *if their union is an orbifold atlas. If* $x \in U_i$, *the* local group Γ_x of $\mathcal{O}$ at a point $x \in \mathcal{O}$ *is the isomorphism class of the stabilizer of the action of* Γ_i *on* $\widetilde{U}_i$ *at the point* $\phi_i^{-1}(x)$. *A point* $x \in \mathcal{O}$ *is* regular *if* Γ_x *is trivial, and* singular *otherwise. The* singular locus *is the set* $\Sigma_{\mathcal{O}}$ *of singular points of* $\mathcal{O}$. *We say that the orbifold* $\mathcal{O}$ *is compact (resp. connected) if the topological space* $|\mathcal{O}|$ *is compact (resp. connected).*

An orientation *for an orbifold atlas for* $\mathcal{O}$ *is given by an orientation on each* $\widetilde{U}_i$ *which is preserved by every change of chart map* ψ *as in part iii) above. The orbifold* $\mathcal{O}$ *is* orientable *if it has an orientation.*

Remark 9.1.2. One can replace metrizable in Definition 9.1.1 by Hausdorff which although is a weaker condition, it suffices for our purposes.

Let $\mathcal{O}$, $\mathcal{O}'$ be smooth orbifolds. An *orbifold diffeomorphism* $f\colon \mathcal{O} \to \mathcal{O}'$ is a homeomorphism at the level of topological spaces $|\mathcal{O}| \to |\mathcal{O}'|$ such that for every $x \in \mathcal{O}$ there are charts $\phi_i\colon \widetilde{U}_i \to U_i$, $x \in U_i$, and $\phi'_j\colon \widetilde{U}'_j \to U'_j$ such that $f(U_i) \subset U'_j$ and the restriction $f|_{U_i}$ may be lifted to a diffeomorphism $\widetilde{f}\colon \widetilde{U}_i \to \widetilde{U}'_j$ which is equivariant with respect to some homomorphism $\Gamma_i \to \Gamma'_j$.

Definition 9.1.3. *An orbifold* $\mathcal{O}$ *is said to be* good *(resp.* very good*) if it is obtained as the quotient of a manifold by a proper action of a discrete (resp. finite) group of diffeomorphisms.*

As the orbifold charts for the orbifold in Definition 9.1.3 we take small neighborhoods of points in the smooth manifold provided with the actions stabilizer subgroups that occur, i.e. the orbifold structure inherited from the manifold structure.

9.1.2. Orbifold connections, orbibundles. To avoid technical problems we do the following definition only for the case we need in the paper.

Definition 9.1.4. *Let* T *be a torus. Let* Y *be a smooth manifold equipped with a smooth effective action of* T*, and let* $\mathcal{O}$ *be a smooth orbifold. A continuous surjective map* $p\colon Y \to \mathcal{O}$ *is a* smooth principal T-orbibundle *if the action of* T *on* Y *preserves the fibers of* p *and acts locally freely and transitively on them, and if for every* $z \in \mathcal{O}$ *the following holds. If* $\{(\widetilde{U}_i, U_i, \phi_i, \Gamma_i)\}_{i\in I}$ *is an orbifold atlas as in Definition 9.1.1, For each* i *there exists a map* $\psi_i\colon T \times \widetilde{U}_i \to Y$ *which induces a* T*-equivariant diffeomorphism between* $T \times_{\Gamma_i} \widetilde{U}_i$*, with the* T*-action on the left factor, and* $p^{-1}(\phi_i(\widetilde{U}_i))$ *such that* $p \circ \psi_i = \phi_i \circ \pi_2$*, where* $\pi_2\colon T \times \widetilde{U}_i \to \widetilde{U}_i$ *is the canonical projection. Here* Γ_i *acts on* $T \times \widetilde{U}_i$ *by the diagonal action, and on* $\widetilde{U}_i$ *by the linearized action.*

A connection *for* $p\colon Y \to \mathcal{O}$ *is a smooth vector subbundle* H *of the tangent bundle* $\mathrm{T}Y$ *with the property that for each* $y \in Y$*,* H_y *is a direct complement in* T_yY *of the tangent space to the fiber of* p *that passes through* y*. We say that the connection* H *is* flat with respect to $p\colon Y \to \mathcal{O}$ *if the subbundle* $H \subset \mathrm{T}Y$ *is integrable considered as a smooth distribution on* Y*.*

Remark 9.1.5. Let $p\colon Y \to \mathcal{O}$ be a smooth principal T-orbibundle as in Definition 9.1.4. Then for every $y \in Y$ there are charts $\phi_i\colon \widetilde{U}_i \to U_i$ of Y, $y \in U_i$, and $\phi'_j\colon \widetilde{U}'_j \to U'_j$ of $\mathcal{O}$, such that $p(U_i) \subset U'_j$ and the restriction $p|_{U_i}$ may be lifted to a smooth map $\widetilde{p}\colon \widetilde{U}_i \to \widetilde{U}'_j$.

If the local group T_y is trivial for all $y \in Y$, the action of T on Y is free and $\mathcal{O}$ is a smooth manifold. Then the mapping $p\colon Y \to \mathcal{O}$ is a smooth principal T-bundle, in the usual sense of the theory of fiber bundles on manifolds, for example c.f. [**52**]. We also use the term T-bundle instead of T-orbibundle.

9.2. Coverings

9.2.1. Lifts, orbifold fundamental group. We start by recalling the notion of orbifold covering.

Definition 9.2.1. [**6**, Sec. 2.2] *A* covering *of a connected orbifold $\mathcal{O}$ is a connected orbifold $\widehat{\mathcal{O}}$ together with a continuous mapping $p\colon \widehat{\mathcal{O}} \to \mathcal{O}$, called an* orbifold covering map*, such that every point $x \in \mathcal{O}$ has an open neighborhood U with the property that for each component V of $p^{-1}(U)$ there is a chart $\phi\colon \widetilde{V} \to V$ of $\widehat{\mathcal{O}}$ such that $p \circ \phi$ is a chart of $\mathcal{O}$. Two coverings $p_1\colon \mathcal{O}_1 \to \mathcal{O}$, $p_2\colon \mathcal{O}_2 \to \mathcal{O}$ are* equivalent *if there exists a diffeomorphism $f\colon \mathcal{O}_1 \to \mathcal{O}_2$ such that $p_2 \circ f = p_1$. A* universal cover *of $\mathcal{O}$ is a covering $p\colon \widehat{\mathcal{O}} \to \mathcal{O}$ such that for every covering $q\colon \widetilde{\mathcal{O}} \to \mathcal{O}$, there exists a unique covering $r\colon \widehat{\mathcal{O}} \to \widetilde{\mathcal{O}}$ such that $q \circ r = p$. The* deck transformation group *of a covering $p\colon \mathcal{O}' \to \mathcal{O}$ is the group of all self-diffeomorphisms $f\colon \mathcal{O}' \to \mathcal{O}'$ such that $p \circ f = p$, and it is denoted by* $\mathrm{Aut}(\mathcal{O}', p)$.

Remark 9.2.2. Assuming the terminology introduced in Definition 9.2.1, the open sets U, V equipped with the restrictions of the charts for $\widehat{\mathcal{O}}$, $\mathcal{O}$, are smooth orbifolds, and the restriction $p\colon V \to U$ is an orbifold diffeomorphism.

Next we define a notion of orbifold fundamental group that extends the classical definition for when the orbifold is a manifold. The first difficulty is to define a “loop in an orbifold”, which we do mostly but not entirely following Boileau-Maillot-Porti [**6**, Sec. 2.2.1]. The definitions for orbifold loop and homotopy of loops in an orbifold which we give are not the most general ones, but give a convenient definition of orbifold fundamental group, which we use in the definition of the model for (M, σ) in Definition 3.3.1.

Definition 9.2.3. *An* orbifold loop $\alpha\colon [0, 1] \to \mathcal{O}$ *in a smooth orbifold $\mathcal{O}$ is represented by:*

- *a continuous map $\alpha\colon [0, 1] \to |\mathcal{O}|$ such that $\alpha(0) = \alpha(1)$ and there are at most finitely many t such that $\alpha(t)$ is a singular point of $\mathcal{O}$, and*
- *for each t such that $\alpha(t)$ is singular, a chart $\phi\colon \widetilde{U} \to U$, $\alpha(t) \in U$, a neighborhood $V(t)$ of t in $[0, 1]$ such that for all $u \in V(t) \setminus \{t\}$, $\alpha(u)$ is regular and lies in U, and a lift $\widetilde{\alpha}|_{V(t)}$ of $\alpha|_{V(t)}$ to $\widetilde{U}$. We say that $\widetilde{\alpha}|_{V(t)}$ is* a local lift of α around t.

We say that two orbifold loops α, $\alpha'\colon [0, 1] \to \mathcal{O}$ respectively equipped with lifts of charts $\widetilde{\alpha}_i$, $\widetilde{\alpha}'_j$ represent the same loop *if the underlying maps are equal* *and the collections of charts satisfy: for each t such that $\alpha'(t) = \alpha(t) \in U_i \cap U'_j$ is singular, where U_i, U'_j are the corresponding charts associated to t, there is a diffeomorphism ψ between a neighborhood of $\widetilde{\alpha}_i(t)$ and a neighborhood of $\widetilde{\alpha}'_j(t)$ such that $\psi(\widetilde{\alpha}_i(s)) = \widetilde{\alpha}'_j(s)$ for all s in a neighborhood of t and $\phi'_j \circ \psi = \phi_i$.*

In Definition 9.2.3, if $\mathcal{O}$ is a smooth manifold, two orbifold loops in $\mathcal{O}$ are *equal* if their underlying maps are equal.

DEFINITION 9.2.4. *Let γ, λ: $[0, 1] \to \mathcal{O}$ be orbifold loops with common initial and end point in a smooth orbifold $\mathcal{O}$ of which the set of singular points has codimension at least 2 in $\mathcal{O}$. We say that γ* is homotopic to λ with fixed end points *if there exists a continuous map $H\colon [0, 1]^2 \to |\mathcal{O}|$ such that*

- *for each (t, s) such that $H(t, s)$ is singular, there is a chart $\phi\colon \widetilde{U} \to U$, $H(t, s) \in U$, a neighborhood $V(t, s)$ of (t, s) in $[0, 1]^2$ such that for all $(u, v) \in V(t, s) \setminus \{(t, s)\}$, $H(u, v)$ is regular and lies in U, and a lift $\widetilde{H}|_{V(t,s)}$ of $H|_{V(t,s)}$ to $\widetilde{U}$ (we call $\widetilde{H}|_{V(t,s)}$* a local homotopy lift of H around (t, s)*),*
- *the orbifold loops $t \mapsto H(t, 0)$ and $t \mapsto H(t, 1)$ from $[0, 1]$ into $\mathcal{O}$ respectively endowed with the local homotopy lifts $\widetilde{H}|_{V(t,0)}$, $\widetilde{H}|_{V(t,1)}$ of H for each $t \in [0, 1]$, are respectively equal to γ and λ as orbifold loops, c.f. Definition 9.2.3, and*
- *H fixes the initial and end point: $H(0, s) = H(1, s) = \gamma(0)$ for all $s \in [0, 1]$.*

The assumption in Definition 9.2.4 on the codimension of the singular locus of the orbifold always holds for symplectic orbifolds, so such requirement is natural in our context, since the orbifold which we will be working with, the orbit space M/T, comes endowed with a symplectic structure. From now on we assume this requirement for all orbifolds.

DEFINITION 9.2.5. [**6**, Sec. 2.2.1] *Let $\mathcal{O}$ be a connected orbifold. The* orbifold fundamental group $\pi_1^{\mathrm{orb}}(\mathcal{O}, x_0)$ of $\mathcal{O}$ based at the point $x_0 \in \mathcal{O}$ *is the set of homotopy classes of orbifold loops with initial and end point point x_0 with the usual composition law by concatenation of loops, as in the classical sense. The set $\pi_1^{\mathrm{orb}}(\mathcal{O}, x_0)$ is a group, and a change of base point results in an isomorphic group which is conjugate by means of a path from one point to another.*

Let $\alpha\colon [0, 1] \to \widehat{\mathcal{O}}$ be an orbifold path in $\widehat{\mathcal{O}}$. The *projection of the path α under a covering mapping $p\colon \widehat{\mathcal{O}} \to \mathcal{O}$*, which we write as $p(\alpha)$, is a path in the orbifold $\mathcal{O}$ whose underlying map is $p \circ \alpha$, and such that for each t for which $(p \circ \alpha)(t)$ is a singular point, if $\alpha(t)$ is regular then there exists a neighborhood V of $\alpha(t)$ such that $p|_V$ is a chart at $(p \circ \alpha)(t)$. This can be used to define the local lift of $p \circ \alpha$ around t; if otherwise $\alpha(t)$ is singular, then by definition of α there is a chart $\phi\colon \widetilde{V} \to V$ at $\alpha(t)$ and a local lift to

$\widetilde{V}$ of α around t, and by choosing $\widetilde{V}$ to be small enough, $\phi \circ p$ is a chart of $(p \circ \alpha)(t)$ giving the local lift of $p \circ \alpha$ around t. We say that α *is a lift of a path* β *in* $\mathcal{O}$ if $p(\alpha) = \beta$. Using the same argument as in the manifold case one can show the following.

LEMMA 9.2.6. *Let T be a torus. Let Y be a smooth manifold equipped with a smooth effective action of T, and let $\mathcal{O}$ be a smooth orbifold. Let H be a connection for a smooth principal T-orbibundle $p\colon Y \to \mathcal{O}$. Let $p_0 \in \mathcal{O}$ and $y_0 = p(p_0)$. Then for any loop $\gamma\colon [0,\,1] \to \mathcal{O}$ in the orbifold $\mathcal{O}$ such that $\gamma(0) = p_0$ there exists a unique horizontal lift $\lambda_\gamma\colon [0,\,1] \to Y$ with respect to the connection H for $p\colon Y \to \mathcal{O}$, such that $\lambda_\gamma(0) = y_0$, where by horizontal we mean that $\mathrm{d}\,\lambda_\gamma(t)/\,\mathrm{d}\,t \in H_{\lambda_\gamma(t)}$ for every $t \in [0,\,1]$.*

9.2.2. Universal covering. It is a theorem of Thurston [**6**, Th. 2.5] that any connected smooth orbifold $\mathcal{O}$ has, up to equivalence, a unique orbifold covering $\widetilde{\mathcal{O}}$ which is universal and whose orbifold fundamental group based at any regular point is trivial. This definition of universal covering extends to smooth orbifolds the classical definition for smooth manifolds. Next we exhibit a construction of $\widetilde{\mathcal{O}}$.

For each $p \in \mathcal{O}$, let $\widetilde{\mathcal{O}}_p$ denote the space of homotopy classes of (orbifold) paths $\gamma\colon [0,\,1] \to \mathcal{O}$ which start at p_0 and end at p. Let $\widetilde{\mathcal{O}}$ denote the set-theoretic disjoint union of the spaces $\widetilde{\mathcal{O}}_p$, where p ranges over the orbit-space $\mathcal{O}$. Let ψ be the set-theoretic mapping $\psi\colon \widetilde{\mathcal{O}} \to \mathcal{O}$ which sends $\widetilde{\mathcal{O}}_p$ to p. As every orbifold, $\mathcal{O}$ has a smooth orbifold atlas $\{(U_i,\, \widetilde{U}_i,\, \phi_i,\, \Gamma_i)\}_{i \in I}$ in which the sets $\widetilde{U}_i$ are simply connected. Pick $p_i \in U_i$, let $\mathcal{F}_i$ be the fiber of ψ over p_i, and let x_i be a point in $\mathcal{F}_i$, for each $i \in I$. Let $x \in \widetilde{U}_i$, $y \in \mathcal{F}_i$, choose $q_i \in \psi_i^{-1}(p_i)$ and choose a path $\widetilde{\lambda}$ in $\widetilde{U}_i$ from q_i to x, and let $\lambda := \phi_i(\widetilde{\lambda})$ equipped with the lift $\widetilde{\lambda}$ at each point, which is an (orbifold) path in U_i such that $\lambda(0) = p_i$. By definition, the element y is a homotopy class of (orbifold) paths from p_0 to p_i. Let γ be a representative of y, and let α be the (orbifold) path obtained by concatenating γ with λ, where α travels along the points in γ first. Since $\widetilde{U}_i$ is simply connected, the homotopy class of α does not depend on the choice of γ and λ, and we define the surjective map $\psi_i(x,\, y) := [\alpha]$ from the Cartesian product $\widetilde{U}_i \times \mathcal{F}_i$ to $\psi^{-1}(U_i)$. Equip each fiber $\mathcal{F}_i$ with the discrete topology and the Cartesian product $\widetilde{U}_i \times \mathcal{F}_i$ with the product topology. Then ψ induces a topology on $\psi^{-1}(U_i)$ whose connected components are the images of sets of the form $\widetilde{U}_i \times \{y\}$ and we set $(\widetilde{U}_i \times \{y\},\, \psi_i|_{\widetilde{U}_i \times \{y\}})$, $y \in \mathcal{F}_i$, $i \in I$, as orbifold charts for an orbifold atlas for $\widetilde{\mathcal{O}}$. The pair $(\widetilde{\mathcal{O}},\, \psi)$ endowed with the equivalence class of atlases of the orbifold atlas for $\widetilde{\mathcal{O}}$ whose orbifold charts are $(\widetilde{U}_i \times \{y\},\, \psi_i|_{\widetilde{U}_i \times \{y\}})$, $y \in \mathcal{F}_i$, $i \in I$, is a regular and universal orbifold covering of the orbifold $\mathcal{O}$ called the *universal cover of* $\mathcal{O}$ *based at* p_0, and denoted simply by $\widetilde{\mathcal{O}}$. The orbifold fundamental group $\pi_1^{\mathrm{orb}}(\widetilde{\mathcal{O}},\, p_0)$ is trivial.

In the paper we apply this construction with $x_0 \in M$, $p_0 = \pi(x_0) \in M/T$. By Proposition 2.3.4 and Remark 2.3.6 the orbit space M/T is a compact connected and smooth $(\dim M - \dim T)$-dimensional orbifold.

9.3. Differential and symplectic forms

A *(smooth) differential form* ω (resp. *symplectic form*) *on the smooth orbifold* $\mathcal{O}$ is given by a collection $\{\widetilde{\omega}_i\}$ where $\widetilde{\omega}_i$ is a Γ_i-invariant differential form (resp. symplectic form) on each $\widetilde{U}_i$ and such that any two of them agree on overlaps: for each $x \in \widetilde{U}_i$ and $y \in \widetilde{U}_j$ with $\phi_i(x) = \phi_j(y)$, there is a diffeomorphism ψ between a neighborhood of x and a neighborhood of y such that $\phi_j(\psi(z)) = \phi_i(z)$ for all z in such neighborhood and $\psi^*\omega_j = \omega_i$. A *symplectic orbifold* is a a smooth orbifold equipped with a symplectic form.

REMARK 9.3.1. Let ω be a differential form on an orbifold $\mathcal{O}$, and suppose that ω is given by a collection $\{\widetilde{\omega}_i\}$ where $\widetilde{\omega}_i$ is a Γ_i-invariant differential form (resp. symplectic form) on each $\widetilde{U}_i$. Because the $\widetilde{\omega}_i$'s which define it are Γ_i-invariant and agree on overlaps, ω is uniquely determined by its values on any orbifold atlas for $\mathcal{O}$ even if the atlas is not maximal.

If $f\colon \mathcal{O}' \to \mathcal{O}$ is an orbifold diffeomorphism, the *pull-back of a differential form* ω *on* $\mathcal{O}$ is the unique differential form ω' on $\mathcal{O}'$ given by $\widetilde{f}^*\widetilde{\omega}_i$ on each chart $\widetilde{f}^{-1}(\widetilde{U}_i)$; we write $\omega' := f^*\omega$. Analogously we define the pullback under a principal T-orbibundle $p\colon Y \to X/T$ as in Definition 9.1.4, of a form ω on X/T, where the maps $\widetilde{p}$ are given in Remark 9.1.5.

If $p\colon \mathcal{O}' \to \mathcal{O}$ is an orbifold covering map, p is a local diffeomorphism in the sense of Remark 9.2.2, and the *pull-back of a differential form* ω *on* $\mathcal{O}$ is the unique differential form ω' on $\mathcal{O}'$ given by $\widetilde{p}^*\widetilde{\omega}_i$ on each chart $\widetilde{p}^{-1}(\widetilde{U}_i)$; we write $\omega' := p^*\omega$.

We say that two symplectic orbifolds $(\mathcal{O}_1, \nu_1)$, $(\mathcal{O}_2, \nu_2)$ are *symplectomorphic* if there is an orbifold diffeomorphism $f\colon \mathcal{O}_1 \to \mathcal{O}_2$ with $f^*\nu_2 = \nu_1$. f is called an *orbifold symplectomorphism*.

We use the notation $\int_{\mathcal{O}} \omega$ for the integral of the differential form ω on the orbifold $\mathcal{O}$. If $(\mathcal{O}, \omega)$ is a symplectic 2-dimensional orbifold, the integral $\int_{\mathcal{O}} \omega$ is known as the *total symplectic area of* $(\mathcal{O}, \omega)$.

9.4. Orbifold homology, Hurewicz map

Following Borzellino's article [**7**, Def. 6] we make the following definition.

DEFINITION 9.4.1. *Let* $\mathcal{O}$ *be a smooth orbifold. The* first integral orbifold homology group $\mathrm{H}_1^{\mathrm{orb}}(\mathcal{O}, \mathbb{Z})$ *of* $\mathcal{O}$ *is defined as the abelianization of the orbifold fundamental group* $\pi_1^{\mathrm{orb}}(\mathcal{O}, x)$ *of* $\mathcal{O}$ *at* x*, where* x *is any point in* $\mathcal{O}$*.*

REMARK 9.4.2. The definition of $\mathrm{H}_1^{\mathrm{orb}}(\mathcal{O}, \mathbb{Z})$ does not depend on the choice of the point $x \in \mathcal{O}$ in the sense that all abelianizations of $\pi_1^{\mathrm{orb}}(\mathcal{O}, x)$, $x \in \mathcal{O}$, are naturally identified with each other.

The Hurewicz map at the level of smooth orbifolds may be defined analogously to the usual Hurewicz map at the level of smooth manifolds, c.f. c.f. Hatcher's [**21**, Sec. 4.2] or Spanier's [**54**, Sec. 7.4]. If $\mathcal{O}$ is a smooth orbifold, the *orbifold Hurewicz map* h_1 is the projection homomorphism from $\pi_1^{\mathrm{orb}}(\mathcal{O}, x_0)$ to its abelianization $\mathrm{H}_1^{\mathrm{orb}}(\mathcal{O}, \mathbb{Z})$.

9.5. Classification of orbisurfaces

Like for compact, connected, orientable smooth surfaces, there exists a classification for compact, connected, orientable smooth orbisurfaces (2-dimensional orbifolds).

Remark 9.5.1. It follows from Definition 9.1.1 that there are only finitely many points in a compact smooth orbifold $\mathcal{O}$ which are singular, so the singular locus $\Sigma_{\mathcal{O}}$ is finite.

The compact, connected, boundaryless, orientable smooth orbisurfaces are classified by the genus of the underlying surface and the n-tuple of cone point orders $(o_k)_{k=1}^n$, where $o_i \leq o_{i+1}$, for all $1 \leq i \leq n-1$, in the sense that the following two statements hold.

Theorem 9.5.2. *First, given a positive integer g and an n-tuple $(o_k)_{k=1}^n$, $o_i \leq o_{i+1}$ of positive integers, there exists a compact, connected, boundaryless, orientable smooth orbisurface $\mathcal{O}$ with underlying topological space a compact, connected surface of genus g and n cone points of respective orders $o_1, \ldots, o_n$. Secondly, let $\mathcal{O}$, $\mathcal{O}'$ be compact, connected, boundaryless, orientable smooth orbisurfaces. Then $\mathcal{O}$ is diffeomorphic to $\mathcal{O}'$ if and only if the genera of their underlying surfaces are the same, and their associated increasingly ordered n-tuples of orders of cone points are equal.*

Proof. Let $\mathcal{O}$, $\mathcal{O}'$ be compact, connected, boundaryless and orientable smooth orbisurfaces which are moreover diffeomorphic. It follows from the definition of diffeomorphism of orbifolds that the genera of their underlying surfaces are the same, and their associated increasingly ordered n-tuples of orders of cone points are equal.

Conversely, let us suppose that $\mathcal{O}$, $\mathcal{O}'$ are boundaryless orientable smooth orbisurfaces with the property that the genera of their underlying surfaces are the same, and their associated increasingly ordered n-tuples of orders of cone points are equal. Write p_k for the cone points of $\mathcal{O}$ ordered so that p_k has order o_k for all k, where $1 \leq k \leq n$. Because $\mathcal{O}$, $\mathcal{O}'$ are compact, connected, boundaryless, orientable and 2-dimensional, for each cone point in $\mathcal{O}$ and each cone point in $\mathcal{O}'$ there exists a neighborhood that is orbifold diffeomorphic to the standard model of the plane modulo a rotation. Therefore, since the tuples of orders of $\mathcal{O}$ and $\mathcal{O}'$ are the same, for each k there is a neighborhood D_k of p_k, a neighborhood D'_k of p'_k and a diffeomorphism $f_k \colon D_k \to D'_k$ such that $f(p_k) = p'_k$. By shrinking each D_k or D'_k if necessary, we may assume that the topological boundaries $\partial_{|\mathcal{O}|}(D_k)$, $\partial_{|\mathcal{O}'|}(D'_k)$ are simple closed curves. The map $f \colon D := \bigcup D_k \to D' := \bigcup D'_k$

defined by $f|_{D_k} := f_k$ is an orbifold diffeomorphism such that $f(p_k) = p'_k$ and $f(D_k) = D'_k$ for all k, where $1 \leq k \leq n$. Since the topological boundaries $C := \partial_{|\mathcal{O}|}(D_k)$, $C := \partial_{|\mathcal{O}'|}(D'_k)$ are simple closed curves, $|\mathcal{O}| \setminus \cup D_k$ and $|\mathcal{O}'| \setminus \cup D'_k$ are surfaces with boundary, and their corresponding boundaries consist of precisely k boundary components, $\partial_{|\mathcal{O}|}(D_1), \dots, \partial_{|\mathcal{O}|}(D_n)$ and $\partial_{|\mathcal{O}'|}(D'_1), \dots, \partial_{|\mathcal{O}'|}(D'_n)$, each of which is a circle. Then by the classification of surfaces with boundary, there exists a diffeomorphism $g \colon C \to C'$. By definition of diffeomorphism, $g(\partial C) = \partial C'$, and hence there exists a permutation τ of $\{1, \dots, n\}$ such that $g(\partial_{|\mathcal{O}|}(D_k)) = \partial_{|\mathcal{O}'|}(D'_{\tau(k)})$ for all k, where $1 \leq k \leq n$. There exist diffeomorphisms of a surface with boundary that permute the boundary circles in any way one wants, and hence by precomposing with an appropriate such diffeomorphism we may assume that τ is the identity. Because of this together with the fact that a diffeomorphism of a circle which preserves orientation is isotopic to the identity map, we can smoothly deform g near the boundary in $\mathcal{O}$ of each D_k so that g agrees with f on $\partial_{|\mathcal{O}|}(C)$. Hence the map $F \colon \mathcal{O} \to \mathcal{O}$ given as $F|_C := g$ and $F|_{\mathcal{O} \setminus C} := f$ is a well defined orbifold diffeomorphism between $\mathcal{O}$ and $\mathcal{O}'$. $\square$

Remark 9.5.3. A geometric classification of orbisurfaces which considers hyperbolic, elliptic and parabolic structures, is given by Thurston in [**58**, Th. 13.3.6] – while such statement is very interesting and complete, the most convenient classification statement for the purpose of this paper is only in differential topological terms, c.f Theorem 9.5.2. The author is grateful to A. Hatcher for providing him with the precise classification statement, and indicating to him how to prove it.

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