Non-Kähler symplectic manifolds with toric symmetries

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Abstract

Drawing on the classification of symplectic manifolds with cosiotropic principal orbits by Duistermaat and Pelayo, in this note we exhibit families of compact symplectic manifolds, such that

(i) no two manifolds in a family are homotopically equivalent,

(ii) each manifold in each family possesses Hamiltonian, and non-Hamiltonian, toric symmetries,

(iii) each manifold has odd first Betti number and hence it is not a Kähler manifold.

This can be viewed as an application of the aforementioned classification.

1 Introduction

To what extent is the category of symplectic manifolds larger than the category of Kähler manifolds?

This question has attracted much interest in symplectic geometry. It was settled by Gompf [13], following the works of Thurston [18], McDuff [14] and others. In particular Gompf showed that every finitely presented group can be realized as the fundamental group of a compact symplectic manifold, hence proving that the category of symplectic manifolds is "much larger" than the category of Kähler manifolds. [1]

Delzant's classification [4] of symplectic $2n$-manifolds admitting a Hamiltonian n-torus action shows that all such manifolds are projective varieties, and hence Kähler (see [6] for a description of Delzant spaces from a complex algebraic geometry viewpoint). Moreover, Karshon [8] proved that symplectic 4-manifolds with Hamiltonian circle actions are Kähler.

Delzant's and Karshon's classifications seemed to suggest that symplectic manifolds with large Hamiltonian symmetry are also Kähler. However, Tolman [19] constructed a 6-dimensional simply connected symplectic manifold with a Hamiltonian 2-torus action with only isolated fixed points that does not admit an invariant Kähler structure. This shows that the category of symplectic manifolds with Hamiltonian symmetry is larger than the category of Kähler manifolds with Hamiltonian symmetry. For other examples see Lerman [10], Lin [11], Karshon [9], Kim [12] and Woodward [20].

Recently, Duistermaat and Pelayo classified symplectic torus actions with coisotropic principal orbits [5, Thms. 9.4, 9.6] and gave a formula for the homology of the manifold (see [3] for a convexity result for symplectic torus actions with coisotropic principal orbits). In this note we observe that many of the manifolds in the Duistermaat-Pelayo classification are non-Kähler by using the homological information; this can be viewed as an application of their classification result (see also [17, Thm. 8.2.1]).

[1] Kähler manifolds have even Betti numbers and there are other restrictions on the fundamental group [1].

Theorem 1.1. *For each $n \geq 2$ there exist infinitely many homotopically inequivalent compact, connected $2n$-manifolds $(M_i,\, \sigma_i)$, $i \in I$, such that:*

(1) M_i is symplectic but not Kähler,

(2) M_i has toric symmetries: M_i admits a Hamiltonian $(n-2)$-torus action, and a commuting free symplectic 2-torus action, such that the principal orbits for the product action are Lagrangian,

(3) M_i is not isomorphic to a product manifold: there does not exist an equivariant symplectomorphism $M_i \to X \times Y$, where X is a symplectic manifold with a Hamiltonian $(n-2)$-torus action, and Y is a symplectic manifold with a free symplectic 2-torus action.

This result follows from a more general, but rather more technical result. One can easily check that for each fixed $n \geq 2$, there are at least $(n-1)\, n/2$ triples of integers

$$\alpha := (k,\, q,\, s)$$

with

$$\begin{aligned}&0 \leq k \leq n-2\\ &n\ \leq q \leq 2n-k-2\\ &1 \leq s \leq n-(q+k)/2\\ &s \text{ odd.}\end{aligned}$$

Then we have:

Lemma 1.2. *For each α there are infinitely many compact connected symplectic $2n$-manifolds $(M_i,\, \sigma_i)$, $i \in I$, such that:*

(i) $\mathrm{b}_1(M_i) = 2n-2k-s$, so M_i is not a Kähler manifold,

(ii) M_i is not homotopically equivalent to M_j if $i \neq j$,

(iii) M_i admits a Hamiltonian k-torus action,

(iv) M_i admits a free symplectic $(q-k)$-torus action, which commutes with the Hamiltonian k-action, and the principal orbits of the product action are coisotropic,

(v) if $k \geq 1$, there does not exist an equivariant [2] symplectomorphism $M_i \to X \times Y$, where X is any symplectic manifold X with a Hamiltonian k-torus action, and Y is any symplectic manifold with a free symplectic $(q-k)$-torus action.

If we take $q = n$, $k = n-2$ and $s = 1$ in Lemma 1.2, we obtain Theorem 1.1.

On the one hand, Lemma 1.2 generalizes the Kodaira-Thurston example [15, Ex. 3.8], which is a compact connected symplectic 4-manifold with a free symplectic 2-torus action and first integral homology

[2] It should be noted that the definition of "equivariant" here does not allow for reparametrizations of the torus. Let T be a torus, and let M, N be symplectic manifolds on which T acts effectively and symplectically. A map $f\colon M \to N$ is equivariant (or T-equivariant) if $f(t \cdot_M x) = t \cdot_N f(x)$. Here $\cdot_M$ and $\cdot_N$ respectively denote the action of T on M and N. Some authors allow for reparametrizations of the torus in this definition, i.e. they say that "f is equivariant" if there is an automorphism $\Lambda\colon T \to T$ such that $f(t \cdot_M x) = \Lambda(t) \cdot_N f(x)$; other authors call such an f "weakly equivariant". In the present paper, we always assume $\Lambda = \mathrm{Id}$.

group $\mathbb{Z}^3$; on the other hand, it provides examples of symplectic manifolds which are not Kähler, but have both Hamiltonian and non-Hamiltonian toric symmetries.

The proof of Lemma 1.2 boils down to a combinatorial analysis of which invariants in the Duistermaat and Pelayo classification lead to families of symplectic manifolds satisfying the given conditions. Some of these invariants can be suitably encoded by interrelated parameters, which explains the large number of them appearing in the proof.

Example 1.3 If KT denotes the Kodaira-Thurston manifold [15, Ex. 3.8], then the Cartesian product $\mathrm{KT} \times S^2$ is a non-Kähler symplectic six-manifold with a free symplectic 2-torus action, and with a commuting Hamiltonian 1-torus action.

Naturally, one might ask if there exist other examples of symplectic manifolds which have the same toric symmetries, and which are not equivariantly symplectomorphic to $\mathrm{KT} \times S^2$. Using the minimal coupling construction, we can manufacture a symplectic structure on a non-trivial fibration $\pi : N \to \mathrm{KT}$ with fibre S^2; moreover, there is a 1-torus which acts on N in a Hamiltonian fashion.

Resorting to the Gysin sequence we see that $\mathrm{H}^1(\mathrm{N}, \mathbb{Z}) \simeq \mathrm{H}^1(\mathrm{KT}, \mathbb{Z}) \simeq \mathbb{Z}^3$. So N does not admit any Kähler structure. However, the minimal coupling construction does not provide us a free symplectic 2-torus action on the symplectic manifold N.

On the other hand, by taking $n = 3$, $k = 1$, $q = 3$ and $s = 1$ in Lemma 1.2 we get an infinite family of non-Kähler symplectic six-manifolds, which have a Hamiltonian 1-torus action, a commuting free symplectic 2-torus action, and are not products. $\oslash$

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2 Consequences of Lemma 1.2

The following iare consequences of Lemma 1.2. If we take $q = 2n-2$, $k = 0$ and $s = 1$ in Lemma 1.2, we obtain Corollary 2.1.

Corollary 2.1. *For each $n \geq 2$ there exist infinitely many homotopically inequivalent compact, connected $2n$-manifolds (M_i, σ_i), $i \in I$, such that:*

(1) M_i is symplectic, and $\mathrm{b}_1(M_i) = 2n-1$, so M_i is not Kähler;

(2) M_i admits a free symplectic $(2n-2)$-torus action, with coisotropic orbits.

The following is a consequence of the proof of Lemma 1.2.

Corollary 2.2. *Suppose that α is an in Lemma 1.2 and that $k \geq 2$. Then for every fixed list of even integers $\lambda_1 > \ldots > \lambda_s > 0$ there exist infinitely many compact connected symplectic $2n$-manifolds M_l, $l \in J$, satisfying properties (iii), (iv) and (v) in Lemma 1.2, and such that*

(1) if $l \neq l'$, M_l is not equivariantly diffeomorphic to $M_{l'}$,

(2) $\mathrm{H}^1(M_l, \mathbb{Z})$ is isomorphic to $\mathbb{Z}^{2n-2k-s} \times \prod_{j=1}^{s}(\mathbb{Z}/\lambda_j\,\mathbb{Z})$ for each $l \in J$.

3 Background on symplectic torus actions

3.1 List of ingredients for a torus

Let T be a torus. As in [5, Sec. 9.2], *a list of ingredients $\mathcal{I}$ for T* consists of the following six ingredients. 1) An antisymmetric bilinear form $\sigma^{\mathfrak{t}}$ on the Lie algebra $\mathfrak{t}$ of T. 2) A subtorus T_{h} of T, of which the Lie algebra $\mathfrak{t}_{\mathrm{h}}$ is contained in $\mathfrak{l} := \ker \sigma^{\mathfrak{t}}$. 3) A Delzant polytope Δ in ${\mathfrak{t}_{\mathrm{h}}}^*$ with center of mass at the origin. 4) A discrete cocompact subgroup P of the additive subgroup $N := (\mathfrak{l}/\mathfrak{t}_{\mathrm{h}})^*$ of $\mathfrak{l}^*$. 5) An antisymmetric bilinear mapping $c : N \times N \to \mathfrak{l}$ with the following two properties. Write $T_{\mathbb{Z}}$ for the kernel of the exponential $\exp : \mathfrak{t} \to T$.

5a) If $\zeta,\, \zeta' \in P,\;\; c(\zeta,\, \zeta') \in T_{\mathbb{Z}}$.

5b) If $\zeta,\, \zeta',\, \zeta'' \in N,\;\; \zeta(c(\zeta',\, \zeta'')) + \zeta'(c(\zeta'',\, \zeta)) + \zeta''(c(\zeta,\, \zeta')) = 0$ (Jacobi Identity).

Finally, ingredient 6), the *holonomy invariant*, is an element $\overline{\tau}$ of the space $\mathcal{T}$ defined by (3.4) as follows.

Definition of the holonomy invariant. We denote by $\mathrm{Hom}_c(P,\, T)$ the space of mappings $\tau : P \to T$, $\zeta \mapsto \tau_\zeta$, such that

$$\tau_{\zeta'}\, \tau_\zeta = \tau_{\zeta+\zeta'}\, \mathrm{e}^{c(\zeta',\, \zeta)/2}, \quad \zeta,\, \zeta' \in P. \tag{3.1}$$

Given an antisymmetric bilinear map $c : N \times N \to \mathfrak{l}$ as in 5), there always exists a map $\tau : P \to T$, $\tau \mapsto \tau_\zeta$, which satisfies condition (3.1). Indeed, let ε^l, $1 \leq l \leq d_N := \dim N$, be a $\mathbb{Z}$-basis of P and let τ_{ε^l} be any element in T. For any $\zeta = \sum_l \zeta_l\, \varepsilon^l \in P$, where $\zeta_l \in \mathbb{Z}$, define

$$\tau_\zeta = \mathrm{e}^{\sum_{l<l'} \zeta_l\, \zeta_{l'}\, c^{l\, l'}/2} \prod_{l=1}^{d_N} (\tau_{\varepsilon^l})^{\zeta_l} \in T, \tag{3.2}$$

where $c^{l\, l'} := c(\varepsilon^l,\, \varepsilon^{l'}) \in \mathfrak{l} \cap T_{\mathbb{Z}}$. Then $\tau : P \to T$ given by (3.2) satisfies (3.1). On the other hand, let $h : P \to T$, $\zeta \mapsto h_\zeta$ be a homomorphism. For every $\tau \in \mathrm{Hom}_c(P,\, T)$, $h \cdot \tau : \zeta \mapsto \tau_\zeta\, h_\zeta$ is in $\mathrm{Hom}_c(P,\, T)$, and the mapping $(h,\, \tau) \mapsto h \cdot \tau$ defines a free, proper, and transitive action of the group of homomorphisms $\mathrm{Hom}(P,\, T)$ on $\mathrm{Hom}_c(P,\, T)$. Since the group $\mathrm{Hom}(P,\, T)$ is a torus with Lie algebra $\mathrm{Hom}(P,\, \mathfrak{t})$ of dimension $\dim N \dim T$, we have that $\mathrm{Hom}_c(P,\, T)$ is diffeomorphic to a torus of dimension $\dim N \dim T$. For each $\zeta' \in N$, $\zeta \mapsto c(\zeta,\, \zeta')$ is an $\mathfrak{l}$-valued homomorphism from N to $\mathfrak{t}$. We write $c(\cdot,\, N)$ for the set of all $c(\cdot,\, \zeta') \in \mathrm{Hom}(P,\, \mathfrak{t})$ with $\zeta' \in N$. The space $c(\cdot,\, N)$ is a linear subspace of $\mathrm{Hom}(P,\, \mathfrak{t})$. We denote by Sym the space of linear mappings $\alpha : \mathfrak{l}^* \to \mathfrak{l}$, $\xi \mapsto \alpha_\xi$, which satisfy the symmetry condition

$$\xi(\alpha_{\xi'}) - \xi'(\alpha_\xi) = 0. \tag{3.3}$$

Let $\alpha \in \mathrm{Sym}$. The restriction $\alpha|_P \colon P \to \mathfrak{l}$ is a homomorphism. The set $\mathrm{Sym}\,|_P$ of all such $\alpha|_P$ is a linear subspace of $\mathrm{Hom}(P,\, \mathfrak{t})$. We define

$$\mathcal{T} := \mathrm{Hom}_c(P,\, T)/\exp \mathcal{A}, \quad \mathcal{A} := c(\cdot,\, N) + \mathrm{Sym}\,|_P, \tag{3.4}$$

as the orbit space of the action of the Lie subgroup $\exp \mathcal{A}$ of $\mathrm{Hom}(P,\, T)$ on $\mathrm{Hom}_c(P,\, T)$. The following is a consequence of [5, Thm. 9.6].

Fact 3.1 (Existence)**.** *Every list of ingredients $\mathcal{I}$ as above gives rise to a compact connected symplectic manifold of dimension* $\dim T + \dim \mathfrak{l}$ *on which T acts symplectically and T_{h} acts Hamiltonianly. If T_{f} is a complementary subtorus to T_{h} in T, T_{f} acts freely on this manifold. Additionally, the principal T-orbits are coisotropic submanifolds.*

3.2 A construction of symplectic manifolds

Following [5] we construct a symplectic manifold as in Fact 3.1. See [5] for proofs.

(a) Definition of the smooth manifold $G \times_H M_{\mathrm{h}}$. Let $c\colon N \times N \to \mathfrak{l}$ be an antisymmetric bilinear mapping as in ingredient 5) of $\mathcal{I}$. Then $\mathfrak{g} := \mathfrak{t} \times N$ equipped with the operation

$$[(X,\,\zeta),\,(X',\,\zeta')] = -(c(\zeta,\,\zeta'),\,0), \quad (X,\,\zeta),\,(X',\,\zeta') \in \mathfrak{g} = \mathfrak{t} \times N,$$

is a 2-step nilpotent Lie algebra, and the operation $(t,\,\zeta)\,(t',\,\zeta') = (t\,t'\,\mathrm{e}^{-c(\zeta,\,\zeta')/2},\,\zeta+\zeta')$ defines a product in $G := T \times N$ for which G is a Lie group with Lie algebra $\mathfrak{g}$. Choose an element $\tau \in \mathrm{Hom}_c(P,\,T)$ such that $\overline{\tau} = (\exp \mathcal{A}) \cdot \tau$, see (3.4). Because the elements $\tau_\zeta \in T$, where $\zeta \in P$, satisfy (3.1), it follows that $H := \{(t,\,\zeta) \in G \mid \zeta \in P \text{ and } t\,\tau_\zeta \in T_{\mathrm{h}}\}$ is a closed Lie subgroup of G and that

$$((t,\,\zeta),\,x) \mapsto (t\,\tau_\zeta) \cdot x : H \times M_{\mathrm{h}} \to M_{\mathrm{h}} \tag{3.5}$$

defines a smooth action of H on the Delzant manifold $(M_{\mathrm{h}},\,\sigma_{\mathrm{h}},\,T_{\mathrm{h}})$ associated to the polytope $\Delta \subset (\mathfrak{t}_{\mathrm{h}})^*$ by Delzant's theorem [4]. The right action of H on G is proper and free because H is a closed Lie subgroup of G, and hence the action of H on $G \times M_{\mathrm{h}}$ defined by $h\,(g,\,x) = (g\,h^{-1},\,h \cdot x)$ is proper and free. The space $G \times_H M_{\mathrm{h}}$ has a unique structure of a smooth manifold for which the canonical projection $\pi : G \times M_{\mathrm{h}} \to G \times_H M_{\mathrm{h}}$ is a principal H-bundle. Since $G \times M_{\mathrm{h}}$ is connected and π is continuous, $G \times_H M_{\mathrm{h}}$ is connected. The projection $(g,\,x) \mapsto g$ induces a G-equivariant smooth fibration $\psi : G \times_H M_{\mathrm{h}} \to G/H$ with fiber M_{h}, the fiber bundle induced from the principal fiber bundle $G \to G/H$ by means of the action of H on M_{h}. Because P is cocompact in N, G/H is compact, and since the fiber M_{h} is compact, $G \times_H M_{\mathrm{h}}$ is compact.

(b) Definition of the symplectic form on $G \times_H M_{\mathrm{h}}$. Let T_{f} be any complementary subtorus to T_{h} in T, and let $\mathfrak{t}_{\mathrm{f}}$ be its Lie algebra. Let $\mu\colon M_{\mathrm{h}} \to \Delta$ be the momentum map of the Hamiltonian T_{h}-action, where recall that we are assuming that the center of mass of Δ is at the origin (so that the momentum map μ is unique). Consider a fixed linear projection $X \in \mathfrak{t} \mapsto X_{\mathfrak{l}} \in \mathfrak{l}$, and let X_{h} denote the $\mathfrak{t}_{\mathrm{h}}$-component of X in the decomposition $\mathfrak{t}_{\mathrm{h}} \oplus \mathfrak{t}_{\mathrm{f}}$. Recall that we choose the torus T_{h} in item 3) such that $\mathfrak{t}_{\mathrm{h}} \subset \mathfrak{l}$, and in this way we have a decomposition $\mathfrak{l} = \mathfrak{t}_{\mathrm{h}} \oplus (\mathfrak{l} \cap \mathfrak{t}_{\mathrm{f}})$. Let c_{h} denote the $\mathfrak{t}_{\mathrm{h}}$-component of c in $\mathfrak{l} = \mathfrak{t}_{\mathrm{h}} \oplus (\mathfrak{l} \cap \mathfrak{t}_{\mathrm{f}})$. Write $\delta a = ((\delta t,\,\delta\zeta),\,\delta x)$ and $\delta' a = ((\delta' t,\,\delta'\zeta),\,\delta' x)$ for two tangent vectors to $G \times M_{\mathrm{h}}$ at $a = ((t,\,\zeta),\,x)$, where we identify each tangent space of the torus T with $\mathfrak{t}$. Write $X = \delta t + c(\delta\zeta,\,\zeta)/2$ and $X' = \delta' t + c(\delta'\zeta,\,\zeta)/2$. Define

$$\begin{aligned} \omega_a(\delta a,\,\delta' a) \;&=\; \sigma^{\mathfrak{t}}(\delta t,\,\delta' t) + \delta\zeta(X'_{\mathfrak{l}}) - \delta'\zeta(X_{\mathfrak{l}}) - \mu(x)(c_{\mathrm{h}}(\delta\zeta,\,\delta'\zeta)) \\ &\quad + (\sigma_{\mathrm{h}})_x(\delta x,\,(X'_{\mathrm{h}})_{M_{\mathrm{h}}}(x)) - (\sigma_{\mathrm{h}})_x(\delta' x,\,(X_{\mathrm{h}})_{M_{\mathrm{h}}}(x)) + (\sigma_{\mathrm{h}})_x(\delta x,\,\delta' x). \end{aligned} \tag{3.6}$$

It follows from [5, Proof of Thm. 9.6] that ω is a basic 2-form for the action of H on $G \times M_{\mathrm{h}}$ and it descends to a symplectic form σ on $G \times_{\mathrm{h}} M_{\mathrm{h}}$.

(c) Definition of the T-action on $G \times_H M_{\mathrm{h}}$. On $G \times M_{\mathrm{h}}$ we have the action of $s \in T$ which sends $((t,\,\zeta),\,x)$ to $((s\,t,\,\zeta),\,x)$. The induced action of T on $G \times_H M_{\mathrm{h}}$ leaves σ invariant. The torus T_{h} acts on $G \times_H M_{\mathrm{h}}$ in a Hamiltonian fashion, c.f. [5, Proof of Thm. 9.6], and the complementary subtorus T_{f} to T_{h} in T acts freely. In [5, Thm. 9.4] Duistermaat and the second author showed the following.

Fact 3.2 (Uniqueness)**.** *Different lists $\mathcal{I}$ of ingredients give rise to non T-equivariantly symplectomorphic symplectic manifolds.*

4 Proof of Lemma 1.2 and Lemma 2.2

Proof of Lemma 1.2. First we prove that for each fixed $n \geq 2$ there are $\frac{(n-1)\,n}{2}$ quadruples of the form $(n,\,k(n),\,q(n),\,1)$ which satisfy the inequalities in the statement of the theorem. Indeed, for any $k(n),\,q(n)$ such that $k(n) \leq n-2$ and $q(n) \leq 2n-k(n)-2$, we have that $q(n)+k(n) \leq (2n-k(n)-2)+k(n) = 2n-2$, hence

$$n - \frac{q(n)+k(n)}{2} \geq 1$$

and therefore $s(n) = 1$ satisfies the last inequality in the statement of the theorem.

On the other hand, there are precisely $n-1$ values that $k(n)$ can take (i.e. $0,\,1,\ldots,n-2$), and if $k(n) = k_0$ then $q(n)$ can take precisely $n-k_0-1$ values. Hence the total number of quadruples $(n,\,k(n),\,q(n),\,1)$ equals $(n-1)+(n-2)+\ldots+2+1$, i.e. $\frac{(n-1)\,n}{2}$.

Fix $n,\,k,\,q,\,s$ as in the statement of the theorem. Take T in Section 3.1 to be the standard q-dimensional torus $(\mathbb{R}/\mathbb{Z})^q$. In this way the Lie algebra $\mathfrak{t}$ of T gets identified with $\mathbb{R}^q$ and $T_{\mathbb{Z}} = \ker(\exp\colon \mathbb{R}^q \to (\mathbb{R}/\mathbb{Z})^q)$ with $\mathbb{Z}^q$. Consider the following list of ingredients $\mathcal{I}$ for T. Let $m := 2n-q$, and notice that $0 < m \leq q$ since by assumption $q \leq 2n-k-2 < 2n$ and $n \leq q$.

Take ingredient 1) to be any antisymmetric bilinear form $\sigma^{\mathfrak{t}}\colon \mathbb{R}^q \times \mathbb{R}^q \to \mathbb{R}$ whose kernel $\mathfrak{l}$ is equal to $\mathbb{R}^m \times \{0_{\mathbb{R}^{q-m}}\} = \mathbb{R}^m$. Write $\mathbb{R}^{q-m} = \{0_{\mathbb{R}^m}\} \times \mathbb{R}^{q-m}$. Since $q+m$ is even, $\frac{q-m}{2}$ is an integer, and positive since $m \leq q$. Such $\sigma^{\mathfrak{t}}$ is then of the form

$$\sum_{i=1}^{\frac{q-m}{2}} v^*_{m+i} \wedge f^*_{m+i}, \tag{4.1}$$

where $v_{m+1},\,f_{m+1},\ldots,v_{m+\frac{q-m}{2}},\,f_{m+\frac{q-m}{2}}$ are a vector basis of $\mathbb{R}^{q-m}$ such that $\sigma^{\mathfrak{t}}(v_{m+i},\,f_{m+j}) = \delta_{ij}$ and

$$\sigma^{\mathfrak{t}}(v_{m+i},\,v_{m+j}) = 0 = \sigma^{\mathfrak{t}}(f_{m+i},\,f_{m+j})$$

for $1 \leq i \leq \frac{q-m}{2}$.

Let ingredient 2) be the subtorus $T_{\mathfrak{h}} = (\mathbb{R}/\mathbb{Z})^k \times \{\bar{0}_{\mathbb{R}^{q-k}}\} = (\mathbb{R}/\mathbb{Z})^k$ of $(\mathbb{R}/\mathbb{Z})^q$. Here we are using that $q \geq n > n-2 \geq k$, and hence $q-k \geq 2$ holds. Moreover, since

$$k+q \leq k+(2n-k-2) = 2n-2 < 2n,$$

we have that $k < 2n-q = m$, and hence there is an inclusion of Lie algebras $\mathfrak{t}_{\mathfrak{h}} = \mathbb{R}^k \times \{0_{\mathbb{R}^{q-k}}\} = \mathbb{R}^k \subset \mathbb{R}^m = \mathfrak{l}$.

Take ingredient 3) to be any Delzant polytope Δ in the dual Lie algebra $(\mathfrak{t}_{\mathfrak{h}})^* = (\mathbb{R}^k)^* = \mathbb{R}^k$ of the Lie algebra $\mathfrak{t}_{\mathfrak{h}}$, with center of mass at the origin. We use the canonical embedding $(\mathbb{R}^m/\mathbb{R}^k)^* \to (\mathbb{R}^m)^*$ to identify $(\mathbb{R}^m/\mathbb{R}^k)^*$ with an $(m-k)$-dimensional subspace of $(\mathbb{R}^m)^*$, and then use the standard inner product $\langle\cdot,\,\cdot\rangle$ on $\mathbb{R}^m$ to identify this subspace with $\{0_{\mathbb{R}^k}\} \times \mathbb{R}^{m-k} = \mathbb{R}^{m-k}$. With this convention we have the following ingredients 4)–6).

Let 4) be the subgroup $P = \mathbb{Z}^{m-k}$ of the additive group $N = \mathbb{R}^{m-k}$, where recall from Section 3.2 that N was defined as $(\mathfrak{l}/\mathfrak{t}_{\mathfrak{h}})^*$. P is a discrete cocompact subgroup of N.

Define ingredient 5) as follows. For a real number z we denote by $[z]$ its integer part and let

$$\Lambda := \{\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_s, \underbrace{\lambda_{s+1} = 0, \cdots, \lambda_{[\frac{m-k}{2}]} = 0}_{[\frac{m-k}{2}]-s}) \;\mid\; \lambda_u \in \mathbb{Z},\ \lambda_u > 0 \ \text{ if } \ 1 \leq u \leq s\}. \tag{4.2}$$

Since by assumption $s \leq n - \frac{q+k}{2}$, we have that $2s \leq 2n - (q+k)$ and hence

$$m - k = (m+q) - (q+k) = 2n - (q+k) \geq 2s.$$

Therefore the right-hand side member of (4.2) is well defined and non-empty. Let $e_1, \ldots, e_{m-k}$ be the canonical basis vectors of $\mathbb{R}^{m-k}$. For any $\lambda = (\lambda_1, \ldots, \lambda_{[\frac{m-k}{2}]}) \in \Lambda$, let $\widehat{c}_\lambda \colon \mathbb{R}^{m-k} \times \mathbb{R}^{m-k} \to \mathbb{R}^{m-k}$ be the mapping defined by

$$\widehat{c}_\lambda(e_i,\, e_j) = \begin{cases} (-1)^{i+1}\, \lambda_{[\frac{i+1}{2}]}\,(e_i + e_j) & \text{if } (i,\, j) \in \{(2\,l-1,\, 2\,l),\, (2\,l,\, 2\,l-1)\}_{l=1}^{s}; \\ 0_{\mathbb{R}^{m-k}} & \text{otherwise.} \end{cases} \tag{4.3}$$

We claim that $\widehat{c}_\lambda$ is an antisymmetric bilinear mapping. It suffices to show that

$$\widehat{c}_\lambda(e_i, e_j) = -\widehat{c}_\lambda(e_j, e_i) \qquad \forall 1 \leq i, j \leq m-k.$$

Indeed, if $i = 2l-1, j = 2l$ for $1 \leq l \leq s$, then

$$\widehat{c}_\lambda(e_{2l-1}, e_{2l}) = \lambda_l\,(e_{2l-1} + e_{2l}) = -(-1)^{2l+1}\lambda_l\,(e_{2l} + e_{2l-1}) = -\widehat{c}_\lambda(e_{2l}, e_{2l-1}).$$

The above argument also proves that $\widehat{c}_\lambda(e_{2l}, e_{2l-1}) = -\widehat{c}_\lambda(e_{2l-1}, e_{2l})$. In any other cases, we have by definition

$$\widehat{c}_\lambda(e_i, e_j) = -\widehat{c}_\lambda(e_j, e_i) = 0.$$

Therefore $\widehat{c}_\lambda$ is an antisymmetric bilinear mapping which maps $\mathbb{Z}^{m-k} \times \mathbb{Z}^{m-k}$ into $\mathbb{Z}^{m-k}$ (notice that we cannot define a non-trivial antisymmetric bilinear form on $\mathbb{R}^{m-k}$, unless m is at least $k+2$). Let $\chi \colon \mathbb{R}^{m-k} \times \mathbb{R}^{m-k} \to \mathbb{R}^k$ be any antisymmetric bilinear form that is $\mathbb{Z}^k$-valued on $\mathbb{Z}^{m-k} \times \mathbb{Z}^{m-k}$. Since χ and $\widehat{c}_\lambda$ are antisymmetric and bilinear, the mapping $c_\lambda \colon \mathbb{R}^{m-k} \times \mathbb{R}^{m-k} \to \mathbb{R}^m$ defined by

$$c_\lambda(e_i,\, e_j) = (\chi(e_i,\, e_j),\;\; \widehat{c}_\lambda(e_i,\, e_j)), \qquad \text{if } i,\, j \in \{1, \ldots, m-k\} \tag{4.4}$$

is antisymmetric and bilinear.

The right hand side of (4.4) satisfies properties 5a) and 5b) in Section 3.1. Indeed, 5a) holds since χ and $\widehat{c}_\lambda$ are respectively $\mathbb{Z}^k$ and $\mathbb{Z}^{m-k}$-valued on $\mathbb{Z}^{m-k} \times \mathbb{Z}^{m-k}$. Since N is identified with $\mathbb{R}^{m-k}$ using the standard inner product $\langle \cdot,\, \cdot \rangle$ and $e_1, \ldots, e_{m-k}$ form a basis of $\mathbb{R}^{m-k}$, property 5b) holds if and only if

$$\langle e_i,\, c_\lambda(e_j,\, e_l) \rangle + \langle e_l,\, c_\lambda(e_i,\, e_j) \rangle + \langle e_j,\, c_\lambda(e_l,\, e_i) \rangle = 0, \quad \text{for every } i,\, j,\, l \in \{1, \ldots, m-k\}, \tag{4.5}$$

where on each of the three summands of the left member of (4.5) we are identifying the e_f on the left side of the inner product $\langle \cdot,\, \cdot \rangle$ with $(0_{\mathbb{R}^k},\, e_f)$ for $f \in \{i,\, j,\, m\}$. Hence expression (4.5) holds if and only if

$$\langle e_i,\, \widehat{c}_\lambda(e_j,\, e_l) \rangle + \langle e_l,\, \widehat{c}_\lambda(e_i,\, e_j) \rangle + \langle e_j,\, \widehat{c}_\lambda(e_l,\, e_i) \rangle = 0, \quad \text{for every } i,\, j,\, l \in \{1, \ldots, m-k\}. \tag{4.6}$$

When $m - k = 2$, expression (4.6) is satisfied for dimensional reasons. Without loss of generality we can assume that $i = 1,\, j = 2$ and $l \geq 3$. It follows from the definition (4.3) of $\widehat{c}_\lambda$ that (4.6) holds. We take c_λ given by (4.4) to be ingredient 5), for $\lambda \in \Lambda$, with Λ as in (4.2).

Finally, we let ingredient 6) be any element

$$[\tau] \in \mathcal{T} := \mathrm{Hom}_{c_\lambda}(\mathbb{Z}^{m-k},\, (\mathbb{R}/\mathbb{Z})^q) / \Big(\exp(c_\lambda(\cdot,\, \mathbb{R}^{m-k}) + \mathrm{Sym}\,|_{\mathbb{Z}^{m-k}}) \Big).$$

By Section 3.1, $\mathcal{T}$ is non-empty.

Define the associated bundle $M_{\mathcal{I}}$, corresponding to the list $\mathcal{I}$ of ingredients above, as

$$M_{\mathcal{I}} := ((\mathbb{R}/\mathbb{Z})^q \times \mathbb{R}^{m-k}) \times_H M_{\rm h}, \tag{4.7}$$

where

$$H := \{(t,\, \xi) \,|\, t \in (\mathbb{R}/\mathbb{Z})^q,\ \xi \in \mathbb{Z}^{m-k},\ t\,\tau_\xi \in (\mathbb{R}/\mathbb{Z})^k\} \leq (\mathbb{R}/\mathbb{Z})^q \times \mathbb{R}^{m-k},$$

$(M_{\rm h},\, T_{\rm h})$ is the Delzant manifold corresponding to the polytope Δ, H is acting on the product $((\mathbb{R}/\mathbb{Z})^q \times \mathbb{R}^{m-k}) \times M_{\rm h}$ by the diagonal action induced by the map (3.5), the action of $(\mathbb{R}/\mathbb{Z})^q$ on $M_{\mathcal{I}}$ is induced by the translational action on the left-most factor of the product $((\mathbb{R}/\mathbb{Z})^q \times \mathbb{R}^{m-k}) \times M_{\rm h}$, and the symplectic form on $M_{\mathcal{I}}$ is defined as the unique smooth form σ such that $\omega = \pi^*\sigma$, where $\pi\colon\, ((\mathbb{R}/\mathbb{Z})^q \times \mathbb{R}^{m-k}) \times M_{\rm h} \to M_{\mathcal{I}}$ is the projection map and ω is the form given by (3.6).

We claim that $M_{\mathcal{I}}$ given by (4.7) is a compact connected $2n$-dimensional symplectic manifold on which the torus $(\mathbb{R}/\mathbb{Z})^q$ acts symplectically, and where the subtorus $T_{\rm h} = (\mathbb{R}/\mathbb{Z})^k$ of $(\mathbb{R}/\mathbb{Z})^q$ acts Hamiltonianly, while any subtorus complementary to $(\mathbb{R}/\mathbb{Z})^k$ in $(\mathbb{R}/\mathbb{Z})^q$ acts freely. Finally, the principal $(\mathbb{R}/\mathbb{Z})^q$-orbits of $M_{\mathcal{I}}$ are coisotropic. These claims follow from Fact 3.1.

Additionally, we claim that there are infinitely many resulting manifolds which are not $(\mathbb{R}/\mathbb{Z})^q$-equivariantly symplectomorphic to a Cartesian product, as stated in the theorem. This claim follows from the following result ([5, Lem. 7.5]), for which we assume the abstract notation of Section 3.2.

For a fixed *choice of complementary subtorus $T_{\rm f}$ to $T_{\rm h}$ in T, $G \times_H M_{\rm h}$ is T-equivariantly symplectomorphic to the Cartesian product of a symplectic $T_{\rm f}$-space $(M_{\rm f},\, \sigma_{\rm f},\, T_{\rm f})$ on which the $T_{\rm f}$-action is free and the Delzant manifold $(M_{\rm h},\, \sigma_{\rm h},\, T_{\rm h})$ if and only if the $\mathfrak{t}_{\rm h}$-component $c_{\rm h}$ of c in the direct sum decomposition $\mathfrak{l} = \mathfrak{t}_{\rm h} \oplus (\mathfrak{l} \cap \mathfrak{t}_{\rm f})$ induced by $T = T_{\rm h}\, T_{\rm f}$ is equal to zero. Here $\mathfrak{t}_{\rm f}$ denotes the Lie algebra of $T_{\rm f}$, and $t \in T$ acts on $M_{\rm f} \times M_{\rm h}$ by sending $(x_{\rm f},\, x_{\rm h})$ to $(t_{\rm f} \cdot x_{\rm f},\, t_{\rm h} \cdot x_{\rm h})$, if $t = t_{\rm f}\, t_{\rm h}$ with $t_{\rm f} \in T_{\rm f}$ and $t_{\rm h} \in T_{\rm h}$.*

In the fact above it may happen that for a certain choice of $T_{\rm f}$ the $\mathfrak{t}_{\rm h}$-component $c_{\rm h}$ of c therein mentioned may be non-trivial, while for a different choice it may be trivial, c.f. [5, Rmk. 7.7].

Fix a complementary subtorus $T_{\rm f}$ to $(\mathbb{R}/\mathbb{Z})^k$ in $(\mathbb{R}/\mathbb{Z})^q$. Assume that $k \geq 1$, and that λ_i is an even number for each i. As before, we are writing $(\mathbb{R}/\mathbb{Z})^k = (\mathbb{R}/\mathbb{Z})^k \times \{\bar{0}_{\mathbb{R}^{q-k}}\}$ for the Hamiltonian torus $T_{\rm h}$, and we identify $\{0_{\mathbb{R}^k}\} \times \mathbb{R}^{m-k} \subset \mathbb{R}^m$ with $\mathbb{R}^{m-k}$ via $(0_{\mathbb{R}^k},\, x) \mapsto x$. Since $m - k \geq 2$, there exists a non-trivial antisymmetric bilinear form $\chi'\colon \mathbb{R}^{m-k} \times \mathbb{R}^{m-k} \to \mathbb{R}^k$ which is $\mathbb{Z}^k$-valued on $\mathbb{Z}^{m-k} \times \mathbb{Z}^{m-k}$, and which is represented by an integral matrix with at least one odd entry.

Choose $\chi := \chi'$, where χ was given in (4.4). Since $(\mathbb{R}/\mathbb{Z})^q = (\mathbb{R}/\mathbb{Z})^k\, T_{\rm f}$, we have the direct sum decomposition $\mathbb{R}^k \oplus (\mathbb{R}^m \cap \mathfrak{t}_{\rm f}) = \mathbb{R}^m$.

Let $\alpha_{\rm f} : \mathbb{R}^{m-k} \to \mathbb{R}^k$ be the inclusion $\iota\colon \mathbb{R}^{m-k} \to \mathbb{R}^m$ followed by the projection $\pi_{\rm f}\colon \mathbb{R}^m \to \mathbb{R}^k$ with respect to $\mathbb{R}^m \cap \mathfrak{t}_{\rm f} \subset \mathbb{R}^m$. We have that the $\mathbb{R}^k$-component

$$(c_\lambda)_{\mathbb{R}^k} = \pi_{\rm f} \circ c_\lambda$$

of c_λ in such a decomposition is non-trivial. Indeed, it follows from (4.4) that

$$(c_\lambda)_{\mathbb{R}^k} = \chi' + \alpha_{\rm f} \circ \widehat{c}_\lambda. \tag{4.8}$$

Since $T_{\rm f}$ is a complementary subtorus to $T_{\rm h}$ in T, we can choose a $\mathbb{Z}$-basis $\{f_1, f_2, \cdots f_k\}$ of $\mathfrak{t}_{\rm h}$, and a $\mathbb{Z}$-basis $\{f_{k+1}, \cdots, f_m, \cdots, f_q\}$ of $\mathfrak{t}_{\rm f}$, such that $\{f_1, f_2, \cdots f_k, f_{k+1}, \cdots, f_q\}$ form a $\mathbb{Z}$-basis of $\mathfrak{t}$, and such that $\mathbb{R}^m \cap \mathfrak{t}_{\rm f} = \text{span}\{f_{k+1}, \cdots, f_m\}$.

Now let $e_1, \cdots, e_{m-k}$ be the canonical basis vectors of $\mathbb{R}^{m-k}$ as in the equation (4.4), and let $\overline{e}_1, \cdots, \overline{e}_k$ be the canonical basis vectors of $\mathbb{R}^k$. We identify $\overline{e}_i$ with $(\overline{e}_i,\, 0_{\mathbb{R}^{m-k}})$ in $\mathbb{R}^m$, and e_i with $(0_{\mathbb{R}^k},\, e_i)$ in $\mathbb{R}^m$. Then $\{\overline{e}_1, \cdots, \overline{e}_k, e_1, \cdots, e_{m-k}\}$ is the canonical basis of $\mathbb{R}^m$; moreover

$$e_i = \sum_{p=1}^{m} \langle e_i,\, f_p^* \rangle\, f_p, \quad \forall\, 1 \leq i \leq m-k,$$

where $\{f_1^*, \cdots, f_p^*, \cdots, f_m^*\}$ is the dual basis of $\{f_1, \cdots, f_p, \cdots, f_m\}$. Since $\{f_1, \cdots, f_m\}$ is an integral basis of $\mathbb{R}^m$, $\langle e_i, f_p^* \rangle$ is an integer for any $1 \leq i \leq m-k$ and $1 \leq p \leq m$. It follows that the matrix of the linear mapping $\alpha_{\mathfrak{f}} \colon \mathbb{R}^{m-k} \to \mathbb{R}^k$ with respect to the basis $\{e_1, \ldots, e_{m-k}\}$ of $\mathbb{R}^{m-k}$ and the basis $\{f_1, \ldots, f_k\}$ of $\mathbb{R}^k$ is integral.

Recall that, by construction, $\hat{c}_\lambda$ is represented by an integral matrix with even entries only. Thus the composition $\alpha_{\mathfrak{f}} \circ \hat{c}_\lambda$ is represented by an integral matrix with even entries only. However, by definition χ' is represented by a matrix with at least one odd entry. Thus, $\chi' + \alpha_{\mathfrak{f}} \circ \hat{c}_\lambda$, the right hand side of (4.8), cannot be represented by a zero matrix. So $(c_\lambda)_{\mathbb{R}^k}$ is non-trivial on $\mathbb{R}^{m-k}$.

Therefore by [5, Lem. 7.5] the resulting manifolds are not $(\mathbb{R}/\mathbb{Z})^q$-equivariantly symplectomorphic to a Cartesian product $(M_{\mathrm{f}},\, T_{\mathrm{f}}) \times (M_{\mathrm{h}},\, (\mathbb{R}/\mathbb{Z})^k)$ of a symplectic manifold with a free T_{f}-action and a symplectic manifold with a Hamiltonian $(\mathbb{R}/\mathbb{Z})^k$-action. Since this proof is independent of the choice of $\mathfrak{t}_{\mathrm{f}}$ (meaning that the same proof works if we choose a different complementary subtorus T_{f}' to T_{f} in T, giving rise to another complentary subspace $\mathfrak{t}_{\mathrm{f}}'$ of $\mathfrak{t}_{\mathrm{h}}$ in $\mathfrak{t}$), it follows that our manifolds (4.7) are not $(\mathbb{R}/\mathbb{Z})^q$-equivariantly symplectomorphic to any such Cartesian product, for any choice T_{f} of a complementary subtorus.

The following was proven in [5, Cor. 8.3], and again we assume the abstract notation of Section 3.2.

Write $T_{\mathrm{f}} := \mathfrak{t}_{\mathrm{f}} \cap T_{\mathbb{Z}}$, let c_{f} denote the $\mathfrak{t}_{\mathrm{f}}$-component of c in the direct sum decomposition $\mathfrak{l} = \mathfrak{t}_{\mathrm{h}} \oplus (\mathfrak{l} \cap \mathfrak{t}_{\mathrm{f}})$, and let Θ denote the additive subgroup of $(T_{\mathrm{f}})_{\mathbb{Z}}$ which is generated by the elements $c_{\mathrm{f}}(\beta,\, \beta')$, such that $\beta,\, \beta' \in P$. The homology group $\mathrm{H}_1(G \times_H M_{\mathrm{h}},\, \mathbb{Z})$ is isomorphic to $((T_{\mathrm{f}})_{\mathbb{Z}}/\Theta) \times P$.

Using this result we next show that infinitely many values of λ lead to homotopically inequivalent spaces. Let $T_{\mathrm{f}} := \{\bar{0}_{\mathbb{R}^k}\} \times (\mathbb{R}/\mathbb{Z})^{q-k} = (\mathbb{R}/\mathbb{Z})^{q-k}$, which is a complementary subtorus to $T_{\mathrm{h}} = (\mathbb{R}/\mathbb{Z})^k$ in $(\mathbb{R}/\mathbb{Z})^q$. On the one hand, by formula (4.2) we have

$$\begin{aligned} ((T_{\mathrm{f}})_{\mathbb{Z}}/\Theta) \times P &= \frac{\mathbb{Z}^{q-k}}{\langle (\lambda_l\, (e_{2\,l-1} + e_{2\,l}),\, 0_{\mathbb{R}^{q-m}}) \;\mid\; l = 1, \ldots, s \rangle} \times \mathbb{Z}^{m-k} && (4.9) \\ &\simeq \frac{\mathbb{Z}^{q-k}}{(\lambda_1\, \mathbb{Z} \times \ldots \times \lambda_s\, \mathbb{Z}) \times \{0\}^{q-k-s}} \times \mathbb{Z}^{m-k} \\ &\simeq (\mathbb{Z}/\lambda_1\, \mathbb{Z}) \times \ldots \times (\mathbb{Z}/\lambda_s\, \mathbb{Z}) \times \mathbb{Z}^{(q-k-s)+(m-k)}. && (4.10) \end{aligned}$$

To go from the left to the right hand side of (4.9) we eliminated the linearly independent generators of the group Θ, which itself is equal to the denominator of the right hand side.

Since by Fact 3.1 we have that $q + m = 2n$, and hence

$$(q-k-s) + (m-k) = 2n - 2k - s,$$

by [5, Cor. 8.3] and expression (4.10), the first homology group of $M_{\mathcal{I}}$ with integer coefficients is isomorphic to

$$(\mathbb{Z}/\lambda_1\, \mathbb{Z}) \times \cdots \times (\mathbb{Z}/\lambda_s\, \mathbb{Z}) \times \mathbb{Z}^{2n-2k-s}. \tag{4.11}$$

In particular, it follows from (4.11) that the first Betti number of $M_{\mathcal{I}}$ is

$$b_1(M_{\mathcal{I}}) = 2n - 2k - s. \tag{4.12}$$

Recall that the parameters $k,\, q,\, s$ are fixed, and let $\alpha := (k,\, q,\, s)$, and $\mathcal{F} := \mathcal{F}_\alpha$ be the family consisting of all the manifolds $M_{\mathcal{I}}$, where $\mathcal{I}$ is a list of ingredients as in the beginning of the proof of the theorem, for varying values of the parameters $\lambda_1 > \ldots > \lambda_s$. For convenience we reindex this family as $\mathcal{F} = \{M_j\}_{j \in I}$. Expression (4.11) depends on $\lambda = (\lambda_1,\, \ldots,\, \lambda_s)$, and different values of λ may give rise to isomorphic integral homology groups (as a consequence of the classification of finitely generated abelian groups), and hence the corresponding manifolds may be homotopically equivalent. To avoid this we consider the relation on $\mathcal{F}$ given by M_i is related to M_j if and only if the torsion subgroup $\mathrm{H}_1(M_i;\, \mathbb{Z})_{\text{tor}}$ is isomorphic to the torsion subgroup $\mathrm{H}_1(M_j;\, \mathbb{Z})_{\text{tor}}$. This relation is an equivalence relation, and as such it gives rise to a partition of of $\mathcal{F}$ into disjoint subsets. We define $\mathcal{F}'$ to be any subfamily of $\mathcal{F}$ which contains precisely one element of each subset of the partition. Because the elements in $\mathcal{F}'$ have different associated free quotient subgroups, they are not homotopically equivalent to each other. Moreover, because $2n - 2k - s$ is fixed, the set $\mathcal{F}'$ is parametrized by the set of isomorphism classes

$$\{[\mathbb{Z}/\lambda_1\mathbb{Z} \times \ldots \times \mathbb{Z}/\lambda_s\mathbb{Z}] \,|\, \lambda_i \in \mathbb{Z},\;\; \lambda_i \text{ even}\}.$$

This set is infinite because it contains the equivalence classes $[\mathbb{Z}/2\mathbb{Z} \times \ldots \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/(2r)\mathbb{Z}],\quad r \geq 1$, which are different for distinct values of r, since $\mathbb{Z}/2\mathbb{Z} \times \ldots \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/(2r)\mathbb{Z}$ has $r2^s$ elements.

Now recall the following well-known result (c.f. Griffiths and Harris [7, p. 117]).

The odd dimensional Betti numbers of a Kähler symplectic manifold are even.

Since by the definition of λ the integer s is odd, it follows from (4.12) that $b_1(M_{\mathcal{I}})$ an odd number. Hence by [7, p. 117], $M_{\mathcal{I}}$ is non-Kähler. The family $\mathcal{F}$ satisfies the properties (i)-(v) in Lemma 1.2, hence concluding its proof.

Proof of Corollary 2.2. Delzant [4] proved that a Delant manifold is equivariantly diffeomorphic to a smooth toric variety. In order to prove Corollary 2.2, recall that two Delzant manifolds on which a torus S acts are S-equivariantly diffeomorphic if and only if they have the same associated fan [6, Thm. 5.3, Lem. 5.5]. This follows from the construction of a Delzant manifold starting from a Delzant polytope given in Delzant's paper [4]. Let us fix any Delzant poytope in $\mathbb{R}^k$, and assume that $k \geq 2$. Since $k \geq 2$, we can increase the number of vertices of the polytope by chopping corners in an equivariant fashion (so that the resulting polytope is still Delzant, see Remark 4.1; i.e. we remove integral simplices attached to the corners [16, Def. 2.5]). This process changes the fan associated to the polytope, and hence the equivariant diffeomorphism type of its associated Delzant manifold. Next we use the following observation, which follows from [5, Proof of Thm. 9.6].

If two Delzant manifolds $(M_{\mathrm{h}},\, T_{\mathrm{h}})$, $(M'_{\mathrm{h}},\, T_{\mathrm{h}})$ are not T_{h}-equivariantly diffeomorphic, then the corresponding model spaces $G \times_H M_{\mathrm{h}}$ and $G \times_H M'_{\mathrm{h}}$ are not T-equivariantly diffeomorphic. (The definition of G', H' is analogous to the definition of G, H, respectively, where everywhere T'_{h} gets replaced by T_{h}.)

Let us assume the notation of the proof of Lemma 1.2. Fix a manifold $M_i := M_{\mathcal{I}}$ satisfying the properties of Lemma 1.2 and whose first integral homology group is given by (4.11). If we change the Delzant polytope (ingredient 2) of $\mathcal{I}$) according to the previous paragraph, and leave the other ingredients in $\mathcal{I}$ fixed, the resulting infinitely many manifolds as in (4.7) are not $(\mathbb{R}/\mathbb{Z})^q$-equivariantly diffeomorphic to each other, and hence for a fixed λ there exist infinitely many not $(\mathbb{R}/\mathbb{Z})^q$-equivariantly-diffeomorphic manifolds, among the ones we have constructed, with first integral homology group isomorphic to $(\mathbb{Z}/\lambda_1\,\mathbb{Z}) \times \ldots \times (\mathbb{Z}/\lambda_s\,\mathbb{Z}) \times \mathbb{Z}^{2n-2k-s}$.

Remark 4.1 The Δ be an n-dimensional Delzant polytope and let M_Δ be the associated $2n$-dimensional Delzant manifold. Let $\mu_\Delta \colon M_\Delta \to \Delta$ be the associated momentum map. Let $\epsilon > 0$ be small. The ϵ-blow up of M_Δ at a fixed point q, with $\mu_\Delta(q) = p$, is a new Delzant manifold whose associated Delzant polytope is obtained from Δ by the replacing the vertex p by the n vertices $p + \epsilon u_i$, $i = 1, \ldots, n$, where $u_1, \ldots, u_n$ are the primitive inward-pointing edge vectors at the vertex p of Δ. In other words, the momentum polytope of the blow-up of M_Δ at q is obtained from Δ by chopping off the corner corresponding to q, hence replacing the vertex p by n new vertices. See for example [2, Thm. I.3.9] for more details. $\oslash$

Remark 4.2 Note that the statement of Corollary 2.2 does not follow immediately from the proof of Lemma 1.2. In Lemma 1.2 we give homotopically inequivalent manifolds by varying the parameters λ_i. In Corollary 2.2 the parameters λ_i are fixed. $\oslash$

References

[1] J. Amors, M. Burger, K. Corlette, D. Kotschick and D. Toledo: Fundamental groups of compact KŁhler manifolds. Mathematical Surveys and Monographs, 44. American Mathematical Society, Providence, RI, 1996. xii+140 pp.

[2] M. Audin, A. Cannas da Silva and E Lerman: Symplectic geometry of integrable Hamiltonian systems. Lectures delivered at the Euro Summer School held in Barcelona, July 10–15, 2001. Advanced Courses in Mathematics. CRM Barcelona. Birkhäuser Verlag, Basel, 2003. x+225 pp.

[3] Y. Benoist: Actions symplectiques de groupes compacts. *Geometriae Dedicata* **89** (2002) 181–245.

[4] T. Delzant: Hamiltoniens périodiques et image convex de l'application moment. *BSMF* **116** (1988) 315–339.

[5] J.J. Duistermaat, A. Pelayo: Symplectic torus actions with coisotropic principal orbits. *Ann. Inst. Fourier* **57** Vol 7 (2007) 2239–2327.

[6] J.J. Duistermaat, A. Pelayo: Reduced phase space coordinatizations and toric variety coordinatizations of Delzant spaces. Math. Proc. Camb. Phil. Soc. **146** (2009) 695–718.

[7] P. Griffiths and J. Harris: *Principles of algebraic geometry*. Reprint of the 1978 original. Wiley Classics Library. John Wiley & sons, Inc, New York, 1994. xiv+813pp.

[8] Y. Karshon: Periodic Hamiltonian flows on four-dimensional manifolds. *Mem. AMS* No. 672 **141** (1999), viii+71 pp.

[9] Y. Karshon: Example of a non-log-concave Duistermaat-Heckman measure, *Math. Research Lett.*, **3** (1996), no.4, 537-540.

[10] E. Lerman: A compact symmetric symplectic non Kaehler manifold, *Math. Research Lett.*, **3** (1996), 587-590.

[11] Y. Lin: Examples of non-Kähler Hamiltonian circle manifolds with the strong Lefschetz property, *Adv. Math.* **208** (2007), 2, 699-709.

[12] M. K. Kim: A compact symmetric symplectic non-Kähler manifold: Revisit, *Math. Res. Lett.*, **9** (2002), 423-424.

[13] R. Gompf: A new construction of symplectic manifolds, *Ann. Math.*, **142** (1995), 527-598.

[14] D. McDuff: Examples of simply connected non Kaehlerian manifolds, *J. Diff. Geom.* **20** (1984), 267-277.

[15] D. McDuff and D. Salamon: *Introduction to Symplectic Topology*, 2nd edition. OUP, 1998.

[16] A. Pelayo: Toric symplectic ball packing, *Top. Appl.* **153** (2006) 3633–3644.

[17] A. Pelayo: Symplectic actions of two-tori on four-manifolds, *Mem. Amer. Math. Soc.*, to appear.

[18] W. Thurston: Some simple examples of symplectic manifolds, *Proc. Amer. Math. Soc.* **55** (1976) 467-468.

[19] S. Tolman: Examples of non-Kähler Hamiltonian torus actions, *Invent. Math.* **131** (1998) 299-310.

[20] C. Woodward: Multiplicity-free Hamiltonian actions need not be Kähler, *Invent. Math.* **131** (1998) 311-319.

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