L^2-COHOMOLOGY AND COMPLETE HAMILTONIAN MANIFOLDS

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ABSTRACT. A classical theorem of Frankel for compact Kähler manifolds states that a Kähler S^1-action is Hamiltonian if and only if it has fixed points. We prove a metatheorem which says that when Hodge theory holds on non-compact manifolds, then Frankel's theorem still holds. Finally, we present several concrete situations in which the assumptions of the metatheorem hold.

1. THE CLASSICAL FRANKEL THEOREM

An S^1-action on a symplectic manifold (M, ω) is *Hamiltonian* if there exists a smooth map, the *momentum map*,

$$\mu \colon M \to (\mathfrak{s}^1)^* \simeq \mathbb{R}$$

into the dual $(\mathfrak{s}^1)^*$ of the Lie algebra $\mathfrak{s}^1 \cong \mathbb{R}$ of S^1, such that

$$\mathbf{i}_{\xi_M}\omega := \omega(\xi_M, \cdot) = \mathbf{d}\mu,$$

for some generator ξ of $\mathfrak{s}^1$, that is, the 1-form $\mathbf{i}_{\xi_M}\omega$ is exact. Here ξ_M is the vector field on M whose flow is given by $\mathbb{R} \times M \ni (t, m) \mapsto e^{\mathrm{i}t\xi} \cdot m \in M$, where the dot denotes the S^1-action on M. If (M, ω) is connected, compact and *Kähler*, the following result of T. Frankel is well-known:

Frankel's Theorem ([Fr59]). *Let M be a compact connected Kähler manifold admitting an S^1-action preserving the Kähler structure. If the S^1-action has fixed points, then the action is Hamiltonian.*

This theorem generalizes in various ways; for example, the S^1-action may be replaced by a G-action, where G is any compact Lie group and the Kähler structure may be weakened to a symplectic structure. The purpose of this paper is to generalize Frankel's theorem to certain noncompact complete Riemannian manifolds. More specifically, we describe a set of hypotheses

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under which the proof in the compact case can be generalized. This relies on the existence of a Hodge decomposition on 1-forms.

2. Hodge Decomposition implies Frankel's Theorem

Let (M,ω) be a symplectic manifold. The triple $(\omega, g, \mathbf{J})$ is a *compatible triple* on (M,ω) if g is a Riemannian metric and $\mathbf{J}$ is an almost complex structure such that $g(\cdot,\cdot) = \omega(\cdot, \mathbf{J}\cdot)$. Denote by $\mathrm{d}V_g$ the measure associated to the Riemannian volume.

Let G be a connected Lie group with Lie algebra $\mathfrak{g}$ acting on M by symplectomorphisms, i.e., diffeomorphisms which preserve the symplectic form. We refer to $(M,\,\omega)$ as a *symplectic G-manifold*. Any element $\xi \in \mathfrak{g}$ generates a vector field ξ_M on M, called the *infinitesimal generator*, given by

$$\xi_M(x) := \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \exp(t\xi)\cdot x.$$

The G-action on (M,ω) is said to be *Hamiltonian* if there exists a smooth equivariant map $\mu\colon M \to \mathfrak{g}^*$, called the *momentum map*, such that for all $\xi \in \mathfrak{g}$ we have

$$\mathbf{i}_{\xi_M}\omega := \omega(\xi_M, \cdot) = \mathbf{d}\langle \mu, \xi\rangle,$$

where $\langle\cdot,\cdot\rangle : \mathfrak{g}^* \times \mathfrak{g} \to \mathbb{R}$ is the duality pairing. For example, if $G \simeq (S^1)^k$, $k \in \mathbb{N}$, is a torus, the existence of such a map μ is equivalent to the exactness of the one-forms $\mathbf{i}_{\xi_M}\omega$ for all $\xi \in \mathfrak{g}$.

In this case the obstruction of the action to being Hamiltonian lies in the first de Rham cohomology group of M. The simplest example of a S^1-Hamiltonian action is rotation of the sphere S^2 about the polar axis. The flow lines of the infinitesimal generator defining this action are the latitude circles.

Denote by L^2_λ the Hilbert space of square integrable functions relative to a given measure $\mathrm{d}\lambda$ on M, and write the associated norm either on functions or 1-forms as $\|\cdot\|_{\mathrm{L}^2_\lambda}$. This measure determines a formal adjoint δ_λ of the de Rham differential. A L^2_λ 1-form ω is called *λ-harmonic* if it is in the common null space of $\mathbf{d}$ and δ_λ.

Theorem 1. *Let G be a compact connected Lie group acting on the symplectic manifold (M,ω), with $(\omega, g, \mathbf{J})$ a G-invariant compatible triple. Suppose, in addition, that $\mathrm{d}\lambda = f\mathrm{d}V_g$ is a G-invariant measure on M, where f is smooth and bounded. Suppose that $\|\xi_M\|_{\mathrm{L}^2_\lambda} < \infty$ for all $\xi \in \mathfrak{g}$. Assume that every smooth closed 1-form ω in L^2_λ decomposes as an L^2_λ-orthogonal sum $\mathbf{d}f + \chi$, where $f, \mathbf{d}f \in \mathrm{L}^2_\lambda$, $\chi \in \mathrm{L}^2_\lambda$ is λ-harmonic, and that each cohomology class in $H^1(M)$ has a unique λ-harmonic representative in L^2_λ. If $\mathbf{J}$ preserves the space of L^2_λ harmonic one-forms and the G-action has fixed points on every connected component, then the action is Hamiltonian.*

Proof. The proof extends Frankel's method [Fr59]. For clarity, we divide the proof into several steps.

Step 1 (Infinitesimal invariance of λ-harmonic 1-forms). We show first that if $\alpha \in \Omega^1(M)$ is harmonic and $\|\alpha\|_{\mathrm{L}^2_\lambda} < \infty$, then $\mathcal{L}_{\xi_M}\alpha = 0$. This is standard in the usual setting, but requires checking here since we have that $\delta_\lambda \alpha = 0$ rather than $\delta\alpha = 0$.

If φ is an isometry of (M, g) and preserves the measure $\mathrm{d}\lambda$, then

$$\varphi^*\left(\langle\!\langle \nu, \rho \rangle\!\rangle \,\mathrm{d}\lambda\right) = \langle\!\langle \varphi^*\nu, \phi^*\rho \rangle\!\rangle \,\mathrm{d}\lambda$$

for any $\nu, \rho \in \Omega^1(M)$, where $\langle\!\langle \nu, \rho \rangle\!\rangle$ dentes the pointwise inner product of ν and ρ on M.

Next, denote by $\Phi : G \times M \to M$ the G-action and $F_t := \Phi_{\exp(t\xi)}$ the flow of ξ_M. Since $\mathbf{d}\alpha = 0$ it follows that

$$\mathbf{d}F_t^*\alpha = F_t^*\mathbf{d}\alpha = 0.$$

In addition, since F_t^* commutes with λ, we also have

$$\delta_\lambda F_t^*\alpha = F_t^*\delta_\lambda\alpha = 0.$$

Hence if α is harmonic, then so is $F_t^*\alpha$.

However, because F_t is isotopic to the identity,

$$[F_t^*\alpha] = F_t^\star[\alpha] = [\alpha]$$

in $\mathrm{H}^1(M, \mathbb{R})$, where $F_t^\star$ is the map on cohomology induced by F_t. This implies that $F_t^*\alpha = \alpha$ since this cohomology class contains only one harmonic representative. Taking the t-derivative yields

$$\mathcal{L}_{\xi_M}\alpha = 0,$$

as required.

Step 2 (Using the existence of fixed points). Define

$$\xi_M^\flat := g(\xi_M, \cdot) \in \Omega^1(M).$$

If $\alpha \in \Omega^1(M)$ is harmonic and $\|\alpha\|_{\mathrm{L}^2_\lambda} < \infty$, it follows from Step 1 that

$$0 = \mathcal{L}_{\xi_M}\alpha = \mathbf{i}_{\xi_M}\mathbf{d}\alpha + \mathbf{d}\mathbf{i}_{\xi_M}\alpha = \mathbf{d}\mathbf{i}_{\xi_M}\alpha.$$

Thus $\alpha(\xi_M)$ is constant on each connected component of M. Now, ξ_M vanishes on the fixed point set of G, and each component of M contains at least one such point. Thus $\alpha(\xi_M) \equiv 0$ on M, whence

$$\left\langle \xi_M^\flat, \alpha \right\rangle_{\mathrm{L}^2_\lambda} = \int_M \alpha(\xi_M)\,\mathrm{d}\lambda = 0$$

for any harmonic one-form α satisfying $\|\alpha\|_{\mathrm{L}^2_\lambda} < \infty$.

Step 3 (Hodge decomposition). Since

$$\mathbf{d}\mathbf{i}_{\xi_M}\omega = \mathcal{L}_{\xi_M}\omega = 0$$

and $\|\mathbf{i}_{\xi_M}\omega\|_{\mathrm{L}^2_\lambda} < \infty$, our hypothesis implies that

$$\mathbf{i}_{\xi_M}\omega = \mathbf{d}f^\xi + \chi^\xi,$$

where $f^\xi \in \mathrm{C}^\infty(M)$, $\chi^\xi \in \Omega^1(M)$ is λ-harmonic and $\|\mathbf{d}f^\xi\|_{\mathrm{L}^2_\lambda}, \|\chi^\xi\|_{\mathrm{L}^2_\lambda} < \infty$.

We now prove that $\chi^\xi = 0$. If $\alpha \in \Omega^1(M)$ is any harmonic one-form with $\|\alpha\|_{\mathrm{L}^2_\lambda} < \infty$, then

$$\langle \mathbf{i}_{\xi_M}\omega, \alpha\rangle_{\mathrm{L}^2_\lambda} = \left\langle \xi_M^\flat, \mathbf{J}\alpha \right\rangle_{\mathrm{L}^2_\lambda} = 0$$

by Step 2 since $\mathbf{J}\alpha$ is harmonic (by the hypotheses of the theorem). In particular, since

$$\xi_M^\flat = \mathbf{J}\mathbf{d}f^\xi + \mathbf{J}\chi^\xi$$

and $\xi_M^\flat$ is also orthogonal to the first term on the right, we conclude that $\chi^\xi = 0$.

The conclusion of this step is

$$\mathbf{i}_{\xi_M}\omega = \mathbf{d}f^\xi$$

for any $\xi \in \mathfrak{g}$; note that both sides of this identity are linear in ξ.

Step 4 (Equivariant momentum map). Using a basis $\{e_1, \ldots, e_r\}$ of $\mathfrak{g}$, we define $\mu : M \to \mathfrak{g}^*$ by

$$\mu^\xi := \xi^1 f^{e_1} + \cdots + \xi^r f^{e_r}, \quad \text{where} \quad \xi = \xi^1 e_1 + \cdots + \xi^r e_.$$

Clearly,

$$\mathbf{i}_{\xi_M}\omega = \mathbf{d}\mu^\xi,$$

so μ is a momentum map of the G-action. Since G is compact, one can average μ in the standard way (see, e.g., [MR03, Theorem 11.5.2]) to obtain an equivariant momentum map. This completes the proof of the theorem. □

3. Applications

We now discuss several different criteria which ensure that the results of the last section can be applied. The first is the classical setting of 'unweighted' L^2 cohomology, which is the cohomology of the standard Hilbert complex of L^2 differential forms on a complete Riemannian manifold. The existence of a strong Kodaira decomposition is known in many instances, and we present a few examples. We then discuss two other criteria, the first by Ahmed and Stroock and the second by Gong and Wang, which allow one to prove a similar strong Kodaira decomposition for forms which are in L^2 relative to some weighted measure. We present some examples to which these criteria apply. Finally, we recall some well-known facts about the Hodge theory on spaces with 'fibered boundary' geometry; these include asymptotically conical spaces, as well as the important classes of ALE/ALF/... gravitational instantons. Many of these spaces admit circle actions.

3.1. **Unweighted L^2 cohomology.**

The nature of the Kodaira decomposition and L^2 Hodge theory on a complete manifold relative to the standard volume form is now classical. An account may be found in de Rham's book [DeR84]; see also [Ca02].

Theorem 2. *If (M^n, g) is a complete Riemannian manifold and $0 \le k \le n$, then the following conditions are equivalent:*

(i) $\mathrm{Im}(\mathbf{d}\delta + \delta\mathbf{d}) = (\mathcal{H}_2^k(M))^\perp$;

(ii) *There is an* L^2*-orthogonal decomposition*

$$\mathrm{L}^2(M, \Lambda^k) = \mathrm{Im}\,\mathbf{d} \oplus \mathrm{Im}\,\delta \oplus \mathcal{H}_2^k(M);$$

(iii) $\mathrm{Im}\,\mathbf{d}$ *and* $\mathrm{Im}\,\delta$ *are closed in* $\mathrm{L}^2(M, \Lambda^k)$;

(iv) *The quotients*

$$\overline{\mathrm{ran}\,\mathbf{d}}/\mathrm{ran}\,\mathbf{d} = 0$$

in $\mathrm{L}^2(M, \Lambda^k)$ *and* $\mathrm{L}^2(M, \Lambda^{n-k})$.

If the smooth form $\alpha \in \Omega^k(M)$ *decomposes as* $\mathbf{d}\beta + \delta_{\mu_g}\gamma + \chi$*, then* $\beta \in \Omega^{k-1}(M)$, $\gamma \in \Omega^{k+1}(M)$*, and* $\chi \in \Omega^k(M) \cap \mathcal{H}_k^2(M)$ *are all smooth.*

Theorems 1 and 2 imply the following result.

Corollary 3. *Let G be a compact Lie group which acts isometrically on (M, ω), a $2n$-dimensional complete connected Kähler manifold, and suppose that any one of the conditions* (i) - (iv) *of Theorem 2 holds. If the infinitesimal generators of the action all lie in* $\mathrm{L}^2_{\omega^n}$ *and if the G-action has fixed points, then it is Hamiltonian.*

The only point to note is that since M is Kähler, the complex structure $\mathbf{J}$ preserves the space of harmonic forms [We08, Cor 4.11, Ch. 5].

3.2. **Examples.** There are many common geometric settings where the result above applies. We recall a few of these here.

Conformally compact manifolds: A complete manifold (M, g) is called conformally compact if M is the interior of a compact manifold with boundary $\bar{M}$ and g can be written as $\rho^{-2}\bar{g}$, where ρ is a defining function for $\partial\bar{M}$ (i.e., $\partial\bar{M} = \{\rho = 0\}$ and $\mathrm{d}\rho \neq 0$ there) and $\bar{g}$ is a metric on $\bar{M}$ which is non-degenerate and smooth up to the boundary. The sectional curvatures of g become isotropic near any point $p \in \partial\bar{M}$, with common value $-|\mathrm{d}\rho|^2_{\bar{g}}$. If this value is constant along the entire boundary, then (M, g) is called asymptotically hyperbolic.

An old well-known result [Ma88] states that if $n = \dim M \neq 3$ (automatic if M is symplectic), then the conditions of Theorem 2 are satisfied when $k = 1$, and hence Corollary 3 holds. There are now much simpler proofs of this result; see [Ca01].

As explained in [Ca01], the conditions of Theorem 2 are invariant under quasi-isometry, which means that we obtain a similar result for any symplectic manifold quasi-isometric to a conformally compact space. This allows us, in particular, to substantially relax the regularity conditions on ρ and $\bar{g}$ in this definition.

There is an interesting generalization of this to the set of complete edge metrics. The geometry here is a bit more intricate; as before, M is the interior of a smooth manifold with boundary. Now, however, the boundary

$\partial\bar{M}$ is assumed to be the total space of a fibration over a compact smooth manifold Y with compact fiber F. We can use local coordinates (x, y, z) near a point of the boundary where x is a boundary defining function, y is a set of coordinates on Y lifted to $\partial\bar{M}$ and then extended inward, and z is a set of functions which restrict to coordinates on each fiber F. A metric g on M is called a *complete edge metric* if in each such local coordinate system it takes the form

$$g = \frac{\mathbf{d}x^2 + \sum a_{0\alpha}(x,y,z)\mathbf{d}x\mathbf{d}y_\alpha + \sum a_{\alpha\beta}a_{\alpha\beta}(x,y,z)\mathbf{d}y_\alpha\mathbf{d}y_\beta}{x^2} + \sum b_{0\mu}\frac{\mathbf{d}x}{x}\mathbf{d}z_\mu + b_{\alpha\mu}\frac{\mathbf{d}y_\alpha}{x}\mathbf{d}z_\mu + b_{\mu\nu}\mathbf{d}z_\mu\mathbf{d}z_\nu.$$

The prototype is the product $X \times F$ where X is a conformally compact manifold and F is a compact smooth manifold, or more generally, a manifold which fibers over a neighborhood of infinity in a conformally compact space X with compact smooth fiber F.

The analytic techniques developed in [Ma91] generalize those in [Ma88] and show that if (M, g) is a space with a complete edge metric, and if $\dim Y \neq 2$, then the Hodge Laplace operator on 1-forms is closed.

Surfaces of revolution: A Riemannian surface (M, g) which admits an isometric S^1 action must be a surface of revolution, hence in polar coordinates,

$$g = \mathbf{d}r^2 + f(r)^2\mathbf{d}\theta^2,$$

where $\theta \in S^1$ and either $f > 0$ on $(0, \infty)$ and is a function of r^2 (i.e., its Taylor expansion near $r = 0$ has only even terms) which vanishes at $r = 0$, or else f is strictly positive on all of $\mathbb{R}$. In the first case, $M \cong \mathbb{R}^2$, while in the second, $M \cong S^1 \times \mathbb{R}$.

The symplectic form is $\omega = f(r)\mathbf{d}r \wedge \mathbf{d}\theta$, so the action is generated by the vector field ∂_θ. Since $\mathbf{i}_{\partial/\partial\theta}\omega = -f(r)\mathbf{d}r$, one of the basic hypotheses becomes

$$\|\mathbf{i}_{\partial_\theta}\omega\|^2_{\mathrm{L}^2_\omega} = \int_0^{2\pi}\int_0^{\infty} f(r)^3\,\mathrm{d}r\,\mathrm{d}\theta = 2\pi\int_0^{\infty} f(r)^3\,\mathrm{d}r < \infty. \tag{1}$$

Proposition 4. [Tr09, Theorem 1.2] *If $M \cong \mathbb{R}^2$ and $f \leq Cr^{-k}$ for some $k > 1/3$, then the range of the Hodge Laplace operator on 1-forms is closed.*

With these hypotheses, we can then apply Corollary 3 as before.

It is worth contrasting Proposition 4 with the well-known criterion of McKean [McK70]. This states that if (M^2, g) is simply connected *and* has Gauss curvature $K_g \leq -1$, then the L^2 spectrum of the Laplacian on functions is contained in $[1/4, \infty)$. The spectrum of the Laplacian on 2-forms is the same, and using a standard Hodge-theoretic argument, the spectrum of the Laplacian on 1-forms is contained in $\{0\} \cup [1/4, \infty)$. Thus this curvature bound would also guarantee the conclusion of Theorem 2. Now,

$$K_g = -f''(r)/f(r) \leq -1$$

is the same as

$$f''(r) \geq f(r).$$

Using this and the initial condition $f(0) = 0$, it is not hard to show that f must grow exponentially as $r \to \infty$, so that (1) cannot hold. In other words, McKean's condition is useless for our purposes.

Of course, if the hypotheses of Proposition 4 hold, then we do not need to apply these Hodge-theoretic arguments since the momentum map of this circle action is given by any function $\mu(r)$ satisfying

$$\mu'(r) = -f(r).$$

Compact stratified spaces: Although it is outside the framework of complete manifolds, there is another class of spaces to which these results may be applied. These are the smoothly stratified spaces with iterated edge metrics. These include, at the simplest level, spaces with isolated conic singularities or simple edge singularities. More general spaces of this type are obtained recursively, by using spaces such as these as cross-sections of cones, and these cones can vary over a smooth base. Hodge theory on such spaces was first considered by Cheeger [Ch79]; the recent papers [ALMP12], [ALMP13] provide an alternate approach and generalize the spaces to allow ones for which it is necessary to impose boundary conditions along the strata. A complete Hodge theory is available, cf. the papers just cited. One important way that such spaces might arise in our setting is if the group G acts symplectically on a compact smooth manifold M', but G commutes with the symplectic action by another group K. Then the action of G descends to the quotient $M = M'/K$, and this latter space typically has precisely the stratified structure and iterated edge metric as described above .

3.3. **Ahmed-Stroock conditions.** Under certain rather weak requirements on the geometry of (M, g) and an auxiliary measure

$$\mathrm{d}\lambda = e^{-U}\mathrm{d}V_g,$$

Ahmed and Stroock [AS00, §6] have proved a Hodge-type decomposition. In the theorem below and the rest of the paper, $\Delta f := \operatorname{div}\nabla f = -\delta\mathbf{d}f$ is the usual Laplacian on functions and $\nabla^2 f := \operatorname{Hess} f$ is the Hessian of f, i.e., the second covariant derivative of f.

Theorem 5 ([AS00])**.** *Assume that (M, g) is complete and*

- $\mathrm{Ric}_g \geq -\kappa_1$*;*
- *the curvature operator is bounded above, i.e., $\langle\langle R\alpha, \alpha\rangle\rangle \leq \kappa_2 \|\alpha\|_{\mathrm{L}^2}$ for all $\alpha \in \Omega^2(M)$, where $\kappa_1, \kappa_2 \geq 0$.*

Suppose further that U is a smooth nonnegative proper function on M which satisfies

- *$\Delta U \leq C(1+U)$ and $\|\nabla U\|^2 \leq C\,\mathrm{e}^{\theta U}$ for some $C < \infty$ and $\theta \in (0,1)$;*

- *$\varepsilon U^{1+\varepsilon} \le 1 + \|\nabla U\|^2$ for some $\varepsilon > 0$;*
- *$\langle\langle v, (\nabla^2 U)(v)\rangle\rangle \ge -B\|v\|^2$ for every $x \in M$ and $v \in \mathrm{T}_x M$, where $B < \infty$.*

Write δ_λ for the adjoint of $\mathbf{d}$ relative to $\mathrm{d}\lambda = e^{-U}\mathrm{d}V_g$ and L^2_λ for the associated Hilbert space. Note that since $U \ge 0$, λ is bounded.

Then

(1) [AS00, Theorem 5.1] *There is a strong Hodge decomposition on 1-forms. In particular, if $\alpha \in \mathrm{L}^2_\lambda\Omega^1 \cap \mathcal{C}^\infty$ is closed, then $\alpha = \mathbf{d}f + \chi$, where $f \in \mathrm{L}^2_\lambda \cap \mathcal{C}^\infty$ and $\chi \in \mathcal{H}^1_\lambda$.*

(2) [AS00, Theorem 6.4] *Each class $[\alpha] \in \mathrm{H}^1(M, \mathbb{R})$ has a unique representative in $\mathcal{H}^1_\lambda$.*

Corollary 6. *Assume that M is symplectic and that $(g, \omega, \mathbf{J})$ are a G-invariant compatible triple, and that U is also G-invariant. If the hypotheses of Theorem 5 all hold, $\mathbf{J}\mathcal{H}^1_\lambda \subset \mathcal{H}^1_\lambda$, and if the G-action has fixed points, then it is Hamiltonian.*

3.4. **Gong-Wang conditions.** There are other conditions, discovered by Gong and Wang, which lead to a strong Hodge decomposition.

Theorem 7 ([GW04])**.** *Let G act on the noncompact symplectic manifold (M, ω), and suppose that $(\omega, g, \mathbf{J})$ is a G-invariant compatible triple. Assume that $\mathrm{d}\lambda = e^V\mathrm{d}V_g$ is also G-invariant and has finite total mass. Suppose finally that*

- $\mathrm{Ric} - \mathrm{Hess}(V) \ge -C\mathrm{Id}$;
- *there exists a positive G-invariant proper function $U \in \mathcal{C}^2(M)$ such that $U + V$ is bounded;*
- $\|\nabla U\| \to \infty$ *as* $U \to \infty$;
- $\limsup_{U\to\infty} \left(\Delta U/\|\nabla U\|^2\right) < 1$.

Then there is a strong Hodge decomposition on $\mathrm{L}^2_\lambda\Omega^1(M)$, as before.

Corollary 8. *With all notation as above, if $\mathbf{J}$ preserves $\mathcal{H}^1_\lambda$, and the G-action has fixed points, then it is Hamiltonian.*

3.5. **Further examples.** There are many interesting types of spaces to which the Ahmed-Stroock and Gong-Wang results can be applied, but which are not covered by the more classical Theorem 2. We describe a few of these here, including spaces with asymptotical cylindrical or asymptotically conic ends or with complete fibered boundary geometry. Amongst these are the asymptotically locally Euclidean (ALE) spaces, as well as the slightly more complicated ALF, ALG, and ALH spaces which arise in the classification of gravitational instantons. (We refer to [HHM] for a description of the geometry of ALE/F/G/H spaces.) We can also handle Joyce's quasi-ALE (QALE) spaces [Jo00] and their more flexible Riemannian analogues, the quasi-asymptotically conic (QAC) spaces of [DM14]. The interest in including all of these spaces is that they seem to be intimately intertwined with symplectic geometry; indeed, many of them arise via hyperKähler reduction.

The obvious idea is to let the function U in Theorem 5 depend only on the radial function r on M. Actually, it is clear that Theorem 5 holds on all of M if and only if it holds on each end (with, say, relative boundary conditions on the compact boundaries), so we can immediately localize to each end. We can also replace g on each end by a perhaps simpler metric which is quasi-isometric to it. The general feature of all these spaces is that the distance function r from a suitably chosen inner boundary has "symbolic decay properties", i.e., successively higher derivatives of r decay increasingly more quickly. Writing $U = r^a$, then we require that

$$\text{i) } \Delta U = a(a-1)r^{a-2}|\nabla r|^2 + ar^{a-1}\Delta r \le C(1+r^a)$$
$$\text{ii) } \epsilon U^{1+\epsilon} = \epsilon r^{a(1+\epsilon)} \le 1 + a^2 r^{2a-2}|\nabla r|^2$$
$$\text{iii) } \nabla^2 U = ar^{a-1}\nabla^2 r + a(a-1)r^{a-2}\mathbf{d}r^2 \ge -B.$$

Recalling that $|\nabla r| = 1$ holds in general, then ii) implies that $a > 2$, while i) shows that Δr must grow slower than r, and finally iii) shows that the level sets $\{r = \text{const.}\}$ have some sort of convexity.

Rather than trying to determine the most general spaces for which these restrictions hold, we explain why they are true for the various examples listed above. For the reasons we have explained (namely, that it suffices to consider a quasi-isometric model), we focus on the simplest models for each of these spaces. In each of the following, we consider one end E of M. In general we can apply our results to manifolds M which decompose into some compact piece K and a finite number of ends

$$E_1, \dots, E_N,$$

each of which is of one of the following types.

Cylindrical ends: Here $E = [0,\infty) \times Y$ where (Y, h) is a compact smooth Riemannian manifold, and r is the linear variable on the first factor. The metric is the product $\mathbf{d}r^2 + h$. We obtain conditions i), ii), iii) directly since $\nabla^2 r = 0$.

Conic ends: Now suppose that $E = [1,\infty) \times Y$ where (Y, h) is again a compact smooth manifold and $r \ge 1$, and the metric is given by $g = \mathbf{d}r^2 + r^2 h$. Then

$$\Delta r = (n-1)/r$$

and

$$\nabla^2 r \ge 0,$$

so, once again, all three conditions hold.

Fibered boundary ends: This is slightly more complicated. Suppose that Z is a compact smooth manifold which is the total space of a fibration $\pi : Z \to Y$ with fiber F. Let h be a metric on Y and suppose that k is a symmetric 2-tensor on Z which restricts to each fiber F to be positive definite and so that $\pi^* h + k$ is positive definite on Z. Then

$$E = [1,\infty) \times Z,$$

and

$$g = \mathbf{d}r^2 + r^2\pi^*h + k.$$

In other words, this metric looks conical in the base (Y) directions and cylindrical in the fiber (F) directions. For the specific cases of such metrics that arise in the gravitational instantons above, Y is the quotient of some S^k by a finite group Γ (typically in $\mathrm{SU}(k+1)$) and F is a torus T^ℓ. The four-dimensional ALF/ALG/ALH spaces correspond to the cases

$$(k, \ell) = (2, 1),\ (1, 2),\ (0, 3).$$

The pair $(k, \ell) = (3, 0)$ is precisely that of ALE spaces.

For each of these, it is a simple computation to check that r has all the required properties.

QALE and QAC ends: The geometry of quasi-asymptotically conic spaces are considerably more difficult to describe in general, and we defer to [Jo00] and [DM14] for detailed descriptions of the geometry. These spaces are slightly more complicated in the sense that while they are essentially conical as $r \to \infty$, the cross-sections $\{r = \text{const.}\}$ are families of smooth spaces which converge to a compact stratified space. This is consistent with the fact that QALE spaces arise as (complex analytic) resolutions of quotients $\mathbb{C}^n/\Gamma$. The basic types of estimates for r and its derivatives are almost the same as above, and so conditions i), ii), and iii) still hold. We refer to the monograph and paper cited above for full details.

Bundles over QAC ends: The final example consists of ends E which are bundles over QAC spaces, and with metrics which do not increase the size of the fibers as $r \to \infty$. This is in perfect analogy to how fibered boundary metrics generalize and fiber over conic metrics. The behavior of the function r on these spaces is similarly benign and these same three conditions hold.

These examples have been given with very little detail (in the last two cases, barely any). The reason for including the, here is because they arise frequently. In particular, the last category, i.e., bundles over QAC (or more specifically, QALE) spaces contain the conjectural picture for the important family of moduli spaces of monopoles on $\mathbb{R}^3$. On none of these spaces is the range of the Laplacian on unweighted 1-forms usually closed, but the Ahmed-Stroock conditions provide an easily applicable way to obtain Hodge decompositions on these spaces.

4. History of the problem: Frankel's Theorem and further results

The first result concerning the relationship between the existence of fixed points and the Hamiltonian character of the action is Frankel's celebrated theorem [Fr59] stating that if the manifold is compact, connected, and Kähler, $G = S^1$, and the symplectic action has fixed points, then it must be Hamiltonian (note that $\mathbf{J}\mathcal{H} \subset \mathcal{H}$ holds, see [We08, Cor 4.11, Ch. 5]). Frankel's work has been very influential: for example, Ono [On84] proved the analogue theorem for compact Lefschetz manifolds and McDuff [Mc88, Proposition 2] has shown that any symplectic circle action on a compact connected symplectic 4-manifold having fixed points is Hamiltonian.

However, this result fails in higher dimensions: McDuff [Mc88, Proposition 1] gave an example of a compact connected symplectic 6-manifold with a symplectic circle action which has nontrivial fixed point set (equal to a union of tori), which is nevertheless not Hamiltonian. If the S^1-action is semi-free (i.e., free off the fixed point set), then Tolman and Weitsman [TW00, Theorem 1] have shown that any symplectic S^1-action on a compact connected symplectic manifold having fixed points is Hamiltonian. Feldman [Fe01, Theorem 1] characterized the obstruction for a symplectic circle action on a compact manifold to be Hamiltonian and deduced the McDuff and Tolman-Weitsman theorems by applying his criterion. He showed that the Todd genus of a manifold admitting a symplectic circle action with isolated fixed points is equal either to 0, in which case the action is non-Hamiltonian, or to 1, in which case the action is Hamiltonian. In addition, any symplectic circle action on a manifold with positive Todd genus is Hamiltonian. For additional results regarding aspherical symplectic manifolds (i.e. $\int_{S^2} f^*\omega = 0$ for any smooth map $f : S^2 \to M$) see [KRT08, Section 8] and [LP95]. As of today, there are no known examples of symplectic S^1-actions on compact connected symplectic manifolds that are not Hamiltonian but have at least one isolated fixed point.

Less is known for higher dimensional Lie groups. Giacobbe [Gi05, Theorem 3.13] proved that a symplectic action of a n-torus on a $2n$-dimensional compact connected symplectic manifold with fixed points is necessarily Hamiltonian; see also [DP07, Corollary 3.9]. If $n = 2$ this result can be checked explicitly from the classification of symplectic 4-manifolds with symplectic 2-torus actions given in [Pe10, Theorem 8.2.1] (since cases 2–5 in the statement of the theorem are shown not to be Hamiltonian; the only non-Kähler cases are given in items 3 and 4 as proved in [DP11, Theorem 1.1]).

If G is a Lie group with Lie algebra $\mathfrak{g}$ acting symplectically on the symplectic manifold (M, ω), the action is said to be *cohomologically free* if the Lie algebra homomorphism

$$\xi \in \mathfrak{g} \mapsto [\mathbf{i}_{\xi_M}\omega] \in \mathrm{H}^1(M, \mathbb{R})$$

is injective; $\mathrm{H}^1(M, \mathbb{R})$ is regarded as an abelian Lie algebra. Ginzburg [Gi92, Proposition 4.2] showed that if a torus $\mathbb{T}^k = (S^1)^k$, $k \in \mathbb{N}$, acts symplectically, then there exist subtori $\mathbb{T}^{k-r}$, $\mathbb{T}^r$ such that $\mathbb{T}^k = \mathbb{T}^r \times \mathbb{T}^{k-r}$, the $\mathbb{T}^r$-action is cohomologically free, and the $\mathbb{T}^{k-r}$-action is Hamiltonian. This homomorphism is the obstruction to the existence of a momentum map: it vanishes if and only if the action admits a momentum map. For compact Lie groups the previous result holds only up to coverings. If G is a compact Lie group, then it is well-known that there is a finite covering

$$\mathbb{T}^k \times K \to G,$$

where K is a semisimple compact Lie group. So there is a symplectic action of $\mathbb{T}^k \times K$ on (M, ω). The K-action is Hamiltonian, since K is semisimple. The previous result applied to $\mathbb{T}^k$ implies that there is a finite covering

$$\mathbb{T}^r \times (\mathbb{T}^{k-r} \times K) \to G$$

such that the $(\mathbb{T}^{k-r} \times K)$-action is Hamiltonian and the $\mathbb{T}^r$-action is cohomologically free; this is [Gi92, Theorem 4.1]. The Lie algebra of $\mathbb{T}^{k-r} \times K$ is $\ker(\xi \mapsto [\mathbf{i}_{\xi_M}\omega])$. (It appears that the argument in [Gi92] implicitly requires M to satisfy the Lefschetz condition or more generally the flux conjecture to hold for M. Thus ultimately it depends on [On06] where the flux conjecture is established in full generality. We thank V. Ginzburg for pointing this out.)

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