Reduced phase space and toric variety coordinatizations of Delzant spaces

By JOHANNES J. DUISTERMAAT† AND ALVARO PELAYO‡

Mathematisch Instituut, Universiteit Utrecht
P.O. Box 80 010, 3508 TA Utrecht, The Netherlands.
e-mail: `J.J.Duistermaat@uu.nl`

AND

University of California–Berkeley, Mathematics Department,
970 Evans Hall # 3840
Berkeley, CA 94720-3840, USA.
e-mail: `apelayo@math.berkeley.edu`

(*Received* 6 *May* 2008; *revised* 11 *August* 2008)

Abstract

In this note we describe the natural coordinatizations of a Delzant space defined as a reduced phase space (symplectic geometry view-point) and give explicit formulas for the coordinate transformations. For each fixed point of the torus action on the Delzant polytope, we have a maximal coordinatization of an open cell in the Delzant space which contains the fixed point. This cell is equal to the domain of definition of one of the natural coordinatizations of the Delzant space as a toric variety (complex algebraic geometry view-point), and we give an explicit formula for the toric variety coordinates in terms of the reduced phase space coordinates. We use considerations in the maximal coordinate neighborhoods to give simple proofs of some of the basic facts about the Delzant space, as a reduced phase space, and as a toric variety. These can be viewed as a first application of the coordinatizations, and serve to make the presentation more self-contained.

1. *Introduction*

Let $(M,\, \sigma)$ be a smooth compact and connected symplectic manifold of dimension $2n$ and let T be a torus which acts effectively on $(M,\, \sigma)$ by means of symplectomorphisms. If the action of T on $(M,\, \sigma)$ is moreover Hamiltonian, then $\dim T \leq n$, and the image of the momentum mapping $\mu_T : M \to \mathfrak{t}^*$ is a convex polytope Δ in the dual space $\mathfrak{t}^*$ of $\mathfrak{t}$, where $\mathfrak{t}$ denotes the Lie algebra of T. In the maximal case when $\dim T = n$, $(M,\, \sigma)$ is called a *Delzant space*.

Delzant [**3**, (*) on p. 323] proved that in this case the polytope Δ is very special, a so-called *Delzant polytope*, of which we recall the definition in Section 2. Furthermore Delzant [**3**, Th. 2.1] proved that two Delzant spaces are T-equivariantly symplectomorphic if and only if their momentum mappings have the same image up to a translation by an element

† Research stimulated by KNAW professorship
‡ Research partially funded by an NSF postdoctoral fellowship

of $\mathfrak{t}^*$. Thirdly Delzant [**3**, pp. 328, 329] proved that for every Delzant polytope Δ there exists a Delzant space such that $\mu_T(M) = \Delta$. This Delzant space is obtained as the *reduced phase space* for a linear Hamiltonian action of a torus N on a symplectic vector space E, at a value λ_N of the momentum mapping of the Hamiltonian N-action, where E, N and λ_N are determined by the Delzant polytope.

Finally Delzant [**3**, Sec. 5] observed that the Delzant polytope gives rise to a fan (= éventail in French), and that the Delzant space with Delzant polytope Δ is T-equivariantly diffeomorphic to the *toric variety* M^{toric} defined by the fan. Here M^{toric} is a complex n-dimensional complex analytic manifold, and the action of the real torus T on M^{toric} has an extension to a complex analytic action on M^{toric} of the complexification $T_{\mathbb{C}}$ of T. In our description in Section 5 of the toric variety M^{toric} we do not use fans. The information, for each vertex v of Δ, which codimension one faces of Δ contain v, already suffices to define M^{toric}.

In this note we show that the construction of the Delzant space M as a reduced phase space leads, for every vertex v of the Delzant polytope, to a natural coordinatization φ_v of a T-invariant open cell M_v in M, where M_v contains the unique fixed point m_v in M of the T-action such that $\mu_T(m_v) = v$. We give an explicit construction of the inverse ψ_v of φ_v, which is a maximal diffeomorphism in the sense of Remark 3·9. The construction of ψ_v originated in an attempt to extend the equivariant symplectic ball embeddings from $(\mathbb{B}^{2n}_r,\, \sigma_0) \subset (\mathbb{C}^n,\, \sigma_0)$ into the Delzant space $(M,\, \sigma)$ in Pelayo [**11**] by maximal equivariant symplectomorphisms from open neighborhoods of the origin in $\mathbb{C}^n$ into the Delzant space $(M,\, \sigma)$. If v and w are two different vertices, then the coordinate transformation $\varphi_w \circ {\varphi_v}^{-1}$ is given by the explicit formulas (4.3), (4.4). This system of coordinates gives a new construction of the symplectic manifold with torus action from the Delzant polytope. After we wrote this paper V. Guillemin informed us that he had also considered the idea of this construction.

Let Σ be the set of all strata of the orbit type stratification of M for the T-action. Then the domain of definition M_v of φ_v is equal to the union of all $S \in \Sigma$ such that the fixed point m_v belongs to the closure of S in M, see Corollary 5·6. The strata $S \in \Sigma$ are also the orbits in the toric variety $M^{\mathrm{toric}} \simeq M$ for the action of the complexification $T_{\mathbb{C}}$ of the real torus T, and the domain of definition M_v of φ_v is equal to the domain of definition of a natural complex analytic $T_{\mathbb{C}}$-equivariant coordinatization Φ_v of a $T_{\mathbb{C}}$-invariant open cell. The diffeomorphism $\Phi_v \circ {\varphi_v}^{-1}$, which sends the reduced phase space coordinates to the toric variety coordinates, maps $U_v := \varphi_v(M_v)$ diffeomorphically onto a complex vector space, and is given by the explicit formulas (5.10).

In the toric variety coordinates the complex structure is the standard one and the coordinate transformations are relatively simple Laurent monomial transformations, whereas the symplectic form is generally given by quite complicated algebraic functions. On the other hand, in the reduced phase space coordinates the symplectic form is the standard one, but the coordinate transformations, and also the complex structure, have a more complicated appearance. While these completely explicit coordinate formulas are the main novelty of the paper, we also use them to reprove many of the known results, leading to an efficient and hopefully attractive exposition of the subject.

Let F denote the set of all d codimension one faces of Δ and, for every vertex v of Δ, let F_v denote the set of all $f \in F$ such that $v \in f$. Note that $\#(F_v) = n$ for every vertex v of Δ. For any sets A and B, let A^B denote the set of all A-valued functions on B. If A is a field and the set B is finite, then A^B is a $\#(B)$-dimensional vector space over

A. One of the technical points in this paper is the efficient organization of proofs and formulas made possible by viewing the Delzant space as a reduction of the vector space $\mathbb{C}^F$, and letting, for each vertex v, the coordinatizations φ_v and Φ_v take their values in $\mathbb{C}^{F_v}$. This leads to a natural projection $\rho_v : \mathbb{C}^F \to \mathbb{C}^{F_v}$ obtained by the restriction of functions on F to $F_v \subset F$. For each vertex v the complex vector space $\mathbb{C}^{F_v}$ is isomorphic to $\mathbb{C}^n$, but the isomorphism depends on an enumeration of F_v, the introduction of which would lead to an unnecessary complication of the combinatorics. Similarly our torus T is isomorphic to $\mathbb{R}^n/\mathbb{Z}^n$, but the isomorphism depends on the choice of a $\mathbb{Z}$-basis of the integral lattice $\mathfrak{t}_{\mathbb{Z}}$ in the Lie algebra $\mathfrak{t}$ of T. As for each vertex v a different $\mathbb{Z}$-basis of $\mathfrak{t}_{\mathbb{Z}}$ appears, we also avoid such a choice, keeping T in its abstract form. We hope and trust that this will not lead to confusion with our main references Delzant [**3**], Audin [**2**] and Guillemin [**8**] about Delzant spaces, where $\mathbb{C}^F$, each $\mathbb{C}^{F_v}$, and T is denoted as $\mathbb{C}^d$, $\mathbb{C}^n$, and $\mathbb{R}^n/\mathbb{Z}^n$, respectively.

The organization of this manuscript is as follows. In Section 2 we review the definition of the reduced phase Delzant space, and introduce the notations which will be convenient for our purposes. In Section 3 we define the reduced phase space coordinatizations. In Section 4 we give explicit formulas for the coordinate transformations and describe the reduced phase space Delzant space as obtained by gluing together bounded open subsets of n-dimensional complex vector spaces with these coordinate transformations as the gluing maps. In Section 5 we review the definition of the toric variety defined by the Delzant polytope, prove that the natural mapping from the reduced phase space to the toric variety is a diffeomorphism, and compare the coordinatizations of Section 3 with the natural coordinatizations of the toric variety. In Section 6 we discuss the de Rham cohomology classes of Kähler forms on the toric manifold, which actually are equal to the de Rham cohomology classes of the symplectic forms of the model Delzant spaces. In Section 7 we present these computations for the two simplest classes of examples, the complex projective spaces and the Hirzebruch surfaces.

2. *The reduced phase space*

Let T be an n-dimensional torus, a compact, connected, commutative n-dimensional real Lie group, with Lie algebra $\mathfrak{t}$. It follows that the exponential mapping $\exp : \mathfrak{t} \to T$ is a surjective homomorphism from the additive Lie group $\mathfrak{t}$ onto T. Furthermore, $\mathfrak{t}_{\mathbb{Z}} := \ker(\exp)$ is a discrete subgroup of $(\mathfrak{t}, +)$ such that the exponential mapping induces an isomorphism from $\mathfrak{t}/\mathfrak{t}_{\mathbb{Z}}$ onto T, which we also denote by exp. Note that $\mathfrak{t}_{\mathbb{Z}}$ is defined in terms of the group T rather than only the Lie algebra $\mathfrak{t}$, but the notation $\mathfrak{t}_{\mathbb{Z}}$ has the advantage over the more precise notation $T_{\mathbb{Z}}$ that it reminds us of the fact it is a subgroup of the additive group $\mathfrak{t}$.

Because $\mathfrak{t}/\mathfrak{t}_{\mathbb{Z}}$ is compact, $\mathfrak{t}_{\mathbb{Z}}$ has a $\mathbb{Z}$-basis which at the same time is an $\mathbb{R}$-basis of $\mathfrak{t}$, and each $\mathbb{Z}$-basis of $\mathfrak{t}_{\mathbb{Z}}$ is an $\mathbb{R}$-basis of $\mathfrak{t}$. Using coordinates with respect to an ordered $\mathbb{Z}$-basis of $\mathfrak{t}_{\mathbb{Z}}$, we obtain a linear isomorphism from $\mathfrak{t}$ onto $\mathbb{R}^n$ which maps $\mathfrak{t}_{\mathbb{Z}}$ onto $\mathbb{Z}^n$, and therefore induces an isomorphism from T onto $\mathbb{R}^n/\mathbb{Z}^n$. For this reason, $\mathfrak{t}_{\mathbb{Z}}$ is called the *integral lattice* in $\mathfrak{t}$. However, because we do not have a preferred $\mathbb{Z}$-basis of $\mathfrak{t}_{\mathbb{Z}}$, we do not write $T = \mathbb{R}^n/\mathbb{Z}^n$.

Let Δ be an n-dimensional convex polytope in $\mathfrak{t}^*$. We denote by F and V the set of all codimension one faces and vertices of Δ, respectively. Note that, as a face is defined as the set of points of the closed convex set on which a given linear functional attains its minimum, see Rockafellar [**12**, p.162], every face of Δ is compact. For every $v \in V$, we write

$$F_v = \{f \in F \mid v \in f\}.$$

Δ is called a *Delzant polytope* if it has the following properties, see Guillemin [**8**, p. 8].

i) For each $f \in F$ there is an $X_f \in \mathfrak{t}_{\mathbb{Z}}$ and $\lambda_f \in \mathbb{R}$ such that the hyperplane which contains f is equal to the set of all $\xi \in \mathfrak{t}^*$ such that $\langle X_f, \xi\rangle + \lambda_f = 0$, and Δ is contained in the set of all $\xi \in \mathfrak{t}^*$ such that $\langle X_f, \xi\rangle + \lambda_f \geq 0$. The vector X_f and constant λ_f are made unique by requiring that they are not an integral multiple of another such vector and constant, respectively.
ii) For every $v \in V$, the X_f with $f \in F_v$ form a $\mathbb{Z}$-basis of the integral lattice $\mathfrak{t}_{\mathbb{Z}}$ in $\mathfrak{t}$.

It follows that

$$\Delta = \{\xi \in \mathfrak{t}^* \mid \langle X_f, \xi\rangle + \lambda_f \geq 0 \quad \text{for every} \quad f \in F\}. \tag{2.1}$$

Also, $\#(F_v) = n$ for every $v \in V$, which already makes the polytope Δ quite special. In the sequel we assume that Δ is a given Delzant polytope in $\mathfrak{t}^*$.

For any $z \in \mathbb{C}^F$ and $f \in F$ we write $z(f) = z_f$, which we view as the coordinate of the vector z with the index f. Let π be the real linear map from $\mathbb{R}^F$ to $\mathfrak{t}$ defined by

$$\pi(t) := \sum_{f\in F} t_f\, X_f, \quad t \in \mathbb{R}^F. \tag{2.2}$$

Because, for any vertex v, the X_f with $f \in F_v$ form a $\mathbb{Z}$-basis of $\mathfrak{t}_{\mathbb{Z}}$ which is also an $\mathbb{R}$-basis of $\mathfrak{t}$, we have $\pi(\mathbb{Z}^F) = \mathfrak{t}_{\mathbb{Z}}$ and $\pi(\mathbb{R}^F) = \mathfrak{t}$. It follows that π induces a surjective homomorphism of Lie groups π' from the torus $\mathbb{R}^F/\mathbb{Z}^F = (\mathbb{R}/\mathbb{Z})^F$ onto $\mathfrak{t}/\mathfrak{t}_{\mathbb{Z}}$, and we have the corresponding surjective homomorphism $\exp\circ\pi'$ from $\mathbb{R}^F/\mathbb{Z}^F$ onto T.

Write $\mathfrak{n} := \ker\pi$, a linear subspace of $\mathbb{R}^F$, and $N = \ker(\exp\circ\pi')$, a compact commutative subgroup of the torus $\mathbb{R}^F/\mathbb{Z}^F$. Actually, N is connected, see Lemma 3·1 below, and therefore isomorphic to $\mathfrak{n}/\mathfrak{n}_{\mathbb{Z}}$, where $\mathfrak{n}_{\mathbb{Z}} := \mathfrak{n} \cap \mathbb{Z}^F$ is the integral lattice in $\mathfrak{n}$ of the torus N. [1]

On the complex vector space $\mathbb{C}^F$ of all complex-valued functions on F we have the action of the torus $\mathbb{R}^F/\mathbb{Z}^F$, where $t \in \mathbb{R}^F/\mathbb{Z}^F$ maps $z \in \mathbb{C}^F$ to the element $t \cdot z \in \mathbb{C}^F$ defined by

$$(t \cdot z)_f = \mathrm{e}^{2\pi\,\mathrm{i}\,t_f}\; z_f, \quad f \in F.$$

The infinitesimal action of $Y \in \mathbb{R}^F = \mathrm{Lie}(\mathbb{R}^F/\mathbb{Z}^F)$ is given by

$$(Y \cdot z)_f = 2\pi\,\mathrm{i}\; Y_f\, z_f,$$

which is a Hamiltonian vector field defined by the function

$$z \mapsto \langle Y, \mu(z)\rangle = \sum_{f\in F} Y_f\, |z_f|^2/2 = \sum_{f\in F} Y_f\, ({x_f}^2 + {y_f}^2)/2, \tag{2.3}$$

and with respect to the symplectic form

$$\sigma := (\mathrm{i}\,/4\pi) \sum_{f\in F} \mathrm{d}z_f \wedge \mathrm{d}\overline{z}_f = (1/2\pi) \sum_{f\in F} \mathrm{d}x_f \wedge \mathrm{d}y_f, \tag{2.4}$$

if $z_f = x_f + \mathrm{i}\, y_f$, with $x_f,\, y_f \in \mathbb{R}$. Here the factor $1/2\pi$ is introduced in order to avoid an integral lattice $(2\pi\,\mathbb{Z})^F$ instead of our $\mathbb{Z}^F$.

[1] We did not find a proof of the connectedness of N in [**3**], [**2**], or [**8**].

Because the right hand side of (2.3) depends linearly on Y, we can view $\mu(z)$ as an element of $(\mathbb{R}^F)^* \simeq \mathbb{R}^F$, with the coordinates

$$\mu(z)_f = |z_f|^2/2 = ({x_f}^2 + {y_f}^2)/2, \quad f \in F. \tag{2.5}$$

In other words, the action of $\mathbb{R}^F/\mathbb{Z}^F$ on $\mathbb{C}^F$ is Hamiltonian, with respect to the symplectic form σ and with momentum mapping $\mu : \mathbb{C}^F \to (\mathrm{Lie}(\mathbb{R}^F/\mathbb{Z}^F))^*$ given by (2.3), or equivalently (2.5).

It follows that the subtorus N of $\mathbb{R}^F/\mathbb{Z}^F$ acts on $\mathbb{C}^F$ in a Hamiltonian fashion, with momentum mapping

$$\mu_N := \iota_{\mathfrak{n}}^* \circ \mu : \mathbb{C}^F \to \mathfrak{n}^*, \tag{2.6}$$

where $\iota_{\mathfrak{n}} : \mathfrak{n} \to \mathbb{R}^F$ denotes the identity viewed as a linear mapping from $\mathfrak{n} \subset \mathbb{R}^F$ to $\mathbb{R}^F$, and its transposed $\iota_{\mathfrak{n}}^* : (\mathbb{R}^F)^* \to \mathfrak{n}^*$ is the map which assigns to each linear form on $\mathbb{R}^F$ its restriction to $\mathfrak{n}$.

Write $\lambda_N = \iota_{\mathfrak{n}}^*(\lambda)$, where λ denotes the element of $(\mathbb{R}^F)^* \simeq \mathbb{R}^F$ with the coordinates λ_f, $f \in F$. It follows from Guillemin [**8**, Th. 1.6 and Th. 1.4] that λ_N is a regular value of μ_N, hence the level set $Z := {\mu_N}^{-1}(\{\lambda_N\})$ of μ_N for the level λ_N is a smooth submanifold of $\mathbb{C}^F$, and that the action of N on Z is proper and free. As a consequence the N-orbit space $M = M_\Delta := Z/N$ is a smooth $2n$-dimensional manifold such that the projection $p : Z \to M$ exhibits Z as a principal N-bundle over M. Moreover, there is a unique symplectic form σ_M on M such that $p^*\sigma_M = {\iota_Z}^*\sigma$, where ι_Z is the identity viewed as a smooth mapping from Z to $\mathbb{C}^F$.

Remark 2·1 Guillemin [**8**] used the momentum mapping $\mu_N - \lambda_N$ instead of μ_N, such that the reduction is taken at the zero level of his momentum mapping. We follow Audin [**2**, Ch. VI, Sec. 3.1] in that we use the momentum mapping μ_N for the N-action, which does not depend on λ, and do the reduction at the level λ_N. ⊘

The symplectic manifold (M, σ_M) is the *Marsden-Weinstein reduction* of the symplectic manifold $(\mathbb{C}^F, \sigma)$ for the Hamiltonian N-action at the level λ_N of the momentum mapping, as defined in Abraham and Marsden [**1**, Sec. 4.3]. On the N-orbit space M, we still have the action of the torus $(\mathbb{R}^F/\mathbb{Z}^F)/N \simeq T$, with momentum mapping $\mu_T : M \to \mathfrak{t}^*$ determined by

$$\pi^* \circ \mu_T \circ p = (\mu - \lambda)|_Z. \tag{2.7}$$

The torus T acts effectively on M and $\mu_T(M) = \Delta$, see Guillemin [**8**, Th. 1.7]. Actually, all these properties of the reduction will also follow in a simple way from our description in Section 3 of Z in term of the coordinates z_f, $f \in F$.

The symplectic manifold M_Δ together with this Hamiltonian T-action is called the *Delzant space defined by* Δ, see Guillemin, [**8**, p. 13]. This proves the existence part [**3**, pp. 328, 329] of Delzant's theory.

3. *The reduced phase space coordinatizations.*

For any $v \in V$, let $\iota_v := {\rho_v}^* : (\mathbb{R}^{F_v})^* \to (\mathbb{R}^F)^*$ denote the transposed of the restriction projection $\rho_v : \mathbb{R}^F \to \mathbb{R}^{F_v}$. If in the usual way we identify $(\mathbb{R}^{F_v})^*$ and $(\mathbb{R}^F)^*$ with $\mathbb{R}^{F_v}$ and $\mathbb{R}^F$, respectively, then $\iota_v : \mathbb{R}^{F_v} \to \mathbb{R}^F$ is the embedding defined by $\iota_v(x)_f = x_f$ if $f \in F_v$ and $\iota_v(x)_{f'} = 0$ if $f' \in F$, $f' \notin F_v$. Because ι_v maps $\mathbb{Z}^{F_v}$ into $\mathbb{Z}^F$ and

$\iota_v(\mathbb{R}^{F_v}) \cap \mathbb{Z}^F = \iota_v(\mathbb{Z}^{F_v})$, it induces an embedding of the n-dimensional torus $\mathbb{R}^{F_v}/\mathbb{Z}^{F_v}$ into $\mathbb{R}^F/\mathbb{Z}^F$, which we also denote by ι_v.

LEMMA 3·1. *With these notations, $\mathbb{R}^F$, $\mathbb{Z}^F$, and $\mathbb{R}^F/\mathbb{Z}^F$ are the direct sum of $\mathfrak{n}$ and $\iota_v(\mathbb{R}^{F_v})$, $\mathfrak{n} \cap \mathbb{Z}^n$ and $\iota_v(\mathbb{Z}^{F_v})$, and N and $\iota_v(\mathbb{R}^{F_v}/\mathbb{Z}^{F_v})$, respectively.*

It follows that N is connected, a torus, with integral lattice equal to $\mathfrak{n} \cap \mathbb{Z}^F$. It also follows that $\pi \circ \iota_v$ is an isomorphism from the torus $\mathbb{R}^{F_v}/\mathbb{Z}^{F_v}$ onto the torus T.

Proof. Let $t \in \mathbb{R}^F$. Because the X_f, $f \in F_v$, form an $\mathbb{R}$-basis of $\mathfrak{t}$, there exists a unique $t^v \in \mathbb{R}^{F_v}$, such that

$$\pi(t) = \sum_{f \in F_v} (t^v)_f \, X_f = \pi(\iota_v(t^v)),$$

that is, $t - \iota_v(t^v) \in \mathfrak{n}$. Moreover, because the X_f, $f \in F_v$, also form a $\mathbb{Z}$-basis of $\mathfrak{t}_{\mathbb{Z}}$, we have that $t^v \in \mathbb{Z}^{F_v}$, and therefore $t - \iota_v(t^v) \in \mathfrak{n} \cap \mathbb{Z}^F$, if $t \in \mathbb{Z}^F$. $\square$

LEMMA 3·2. *We have $z \in Z$ if and only if $\mu(z) - \lambda \in \pi^*(\mathfrak{t}^*)$. More explicitly, if and only if there exists a $\xi \in \mathfrak{t}^*$ such that*

$$|z_f|^2/2 - \lambda_f = \langle X_f, \, \xi \rangle \quad \text{for every} \quad f \in F. \tag{3.1}$$

When $z \in Z$, the ξ in (3.1) is uniquely determined.

Furthermore, $Z = (\mu - \lambda)^{-1}(\pi^*(\Delta))$, $(\mu - \lambda)(Z) = \pi^*(\Delta)$, and Z is a compact subset of $\mathbb{C}^F$.

Proof. The kernel of $\iota_{\mathfrak{n}}^*$ is equal to the space of all linear forms on $\mathbb{R}^F$ which vanish on $\mathfrak{n} := \ker \pi$, and therefore $\ker \iota_{\mathfrak{n}}^*$ is equal to the image of $\pi^* : \mathfrak{t}^* \to (\mathbb{R}^F)^*$. Because π is surjective, π^* is injective, which proves the uniqueness of ξ.

It follows from (3.1) that $\langle X_f, \, \xi \rangle + \lambda_f \geq 0$ for every $f \in F$, and therefore $\xi \in \Delta$ in view of (2.1). Conversely, if $\xi \in \Delta$, then there exists for every $f \in F$ a complex number z_f such that $|z_f|^2/2 = \langle X_f, \, \xi \rangle + \lambda_f$, which means that $z \in Z$ and $(\mu - \lambda)(z) = \pi^*(\xi)$. The set $\pi^*(\Delta)$ is compact because Δ is compact and π^* is continuous. Because the mapping $\mu - \lambda$ is proper, it follows that $Z = (\mu - \lambda)^{-1}(\pi^*(\Delta))$ is compact. $\square$

Let $v \in V$. The X_f, $f \in F_v$, form an $\mathbb{R}$-basis of $\mathfrak{t}$, and therefore there exists for each $z \in \mathbb{C}^{F_v}$ a unique $\xi = \mu_v(z) \in \mathfrak{t}^*$ such that (3.1) holds for every $f \in F_v$. That is, the mapping $\mu_v : \mathbb{C}^{F_v} \to \mathfrak{t}^*$ is defined by the equations

$$|z_f|^2/2 - \lambda_f = \langle X_f, \, \mu_v(z) \rangle, \quad z \in \mathbb{C}^{F_v}, \quad f \in F_v. \tag{3.2}$$

In other words, μ_v is defined by the formula

$$\rho_v \circ \pi^* \circ \mu_v = \rho_v \circ (\mu - \lambda) \circ \iota_v, \tag{3.3}$$

where ρ_v denotes the restriction projection from $\mathbb{R}^F$ onto $\mathbb{R}^{F_v}$.

LEMMA 3·3. *If we let T act on $\mathbb{C}^{F_v}$ via $\mathbb{R}^{F_v}/\mathbb{Z}^{F_v}$ by means of $(t, z) \mapsto (\pi \circ \iota_v)^{-1}(t) \cdot z$, then $\mu_v : \mathbb{C}^{F_v} \to \mathfrak{t}^*$ is a momentum mapping for this Hamiltonian action of T on $\mathbb{C}^{F_v}$, with $\mu_v(0) = v$. Here the symplectic form on $\mathbb{C}^{F_v}$ is equal to*

$$\sigma := (\mathrm{i}/4\pi) \sum_{f \in F_v} \mathrm{d}z_f \wedge \mathrm{d}\overline{z}_f = (1/2\pi) \sum_{f \in F_v} \mathrm{d}x_f \wedge \mathrm{d}y_f, \tag{3.4}$$

that is, (2.4) with F replaced by F_v.

Let ρ_v denote the restriction projection from $\mathbb{C}^F$ onto $\mathbb{C}^{F_v}$, and let U_v be the interior of the subset $\rho_v(Z)$ of $\mathbb{C}^{F_v}$. Write

$$\Delta_v := \Delta \setminus \bigcup_{f' \in F \setminus F_v} f'. \tag{3.5}$$

Then $\rho_v(Z) = {\mu_v}^{-1}(\Delta)$, $\mu_v(\rho_v(Z)) = \Delta$, $U_v = {\mu_v}^{-1}(\Delta_v)$, and $\mu_v(U_v) = \Delta_v$. In particular $\rho_v(Z)$ is a compact subset of $\mathbb{C}^{F_v}$, and U_v is a bounded and connected open neighborhood of 0 in $\mathbb{C}^{F_v}$.

Proof. The first statement follows from (3.3), the fact that $\rho_v \circ \mu \circ \iota_v$ is a momentum mapping for the standard $\mathbb{R}^{F_v}/\mathbb{Z}^{F_v}$ action on $\mathbb{C}^{F_v}$, and the fact that a momentum mapping for a Hamiltonian action plus a constant is a momentum mapping for the same Hamiltonian action. It follows in view of (3.2) that $\langle X_f,\, \mu_v(0)\rangle + \lambda_f = 0$ for every $f \in F_v$, hence $\mu_v(0) = v$ in view of i) in the definition of a Delzant polytope, and the fact that $\{v\}$ is the intersection of all the $f \in F_v$.

It follows from (3.2), Lemma 3·2, that $z \in Z$ if and only if

$$|z_f|^2/2 = \langle X_f,\, \mu_v(\rho_v(z))\rangle + \lambda_f \quad \text{for every} \quad f \in F, \tag{3.6}$$

where we note that these equations are satisfied by definition for the $f \in F_v$. Therefore, if $z \in Z$, then (3.6) and (2.1) imply that $\mu_v(\rho(z)) \in \Delta$. Conversely, if $\xi \in \Delta$, then it follows from Lemma 3·2 that there exists $z \in Z$ such that $\pi^*(\xi) = \mu(z) - \lambda$, of which the restriction to F_v yields $\xi = \mu_v(\rho(z))$.

If $\xi \in \Delta_v$, $z^v \in \mathbb{C}^{F_v}$, $\mu_v(z^v) = \xi$, then $\langle X_{f'},\, \mu_v(z^v)\rangle + \lambda_{f'} > 0$ for every $f' \in F \setminus F_v$, which will remain valid if we replace z^v by $\widetilde{z}^v$ in a sufficiently small neighborhood of z^v in $\mathbb{C}^{F_v}$. It follows that we can find $\widetilde{z} \in \mathbb{C}^F$ such that $\rho_v(\widetilde{z}) = \widetilde{z}^v$ and (3.6) holds with z replaced by $\widetilde{z}$. That is, $\widetilde{z} \in Z$, and we have proved that $z^v \in U_v$.

Let conversely $z \in U_v \subset \mathbb{C}^{F_v}$. We have in view of (3.2) that

$$|z_f|^2/2 = \langle X_f,\, \mu_v(z) - \mu_v(0)\rangle = \langle X_f,\, \mu_v(z) - v\rangle$$

for every $f \in F_v$. Therefore $\mu_v(z) - v$ is multiplied by c^2 if we replace z by $c\,z$, $c > 0$. Because z is in the interior of $\rho_v(Z)$, we have $c\,z^v \in \rho_v(Z)$, hence $\mu_v(c\,z) \in \Delta$ for $c > 1$, c sufficiently close to 1. On the other hand, if ξ belongs to a face of Δ which is not adjacent to v, then $v + \tau\,(\xi - v) \notin \Delta$ for any $\tau > 1$. It follows that $\mu_v(z)$ does not belong to any $f' \in F \setminus F_v$, that is, $\mu_v(z) \in \Delta_v$. $\square$

The equation (3.6) can be written in the form $|z_f| = r_f(\mu_v(\rho_v(z)))$, where, for each $f \in F$, the function $r_f : \Delta \to \mathbb{R}_{\geq 0}$ is defined by

$$r_f(\xi) := (2(\langle X_f,\, \xi\rangle + \lambda_f))^{1/2}, \quad f \in F,\ \xi \in \Delta. \tag{3.7}$$

We now view the equations (3.6) for $z \in Z$ as equations for the coordinates $z_{f'}$, $f' \in F \setminus F_v$, with the z_f, $f \in F_v$ as parameters, where the latter constitute the vector $z^v = \rho_v(z)$. If $z^v \in U_v$, then for each $f' \in F \setminus F_v$ the coordinate $z_{f'}$ lies on the circle about the origin with *strictly positive* radius $r_{f'}(\mu_v(z^v))$. Lemma 3·1 implies that the homomorphism which assigns to each element of N its projection to $\mathbb{R}^{F \setminus F_v}/\mathbb{Z}^{F \setminus F_v}$ is an isomorphism, and the latter torus is the group of the coordinatewise rotations of the $z_{f'}$, $f' \in F \setminus F_v$.

On the other hand, if we let $Z_v := {\rho_v}^{-1}(U_v) \cap Z$, which is an open subset of $Z \subset \mathbb{C}^F$, the differential of μ_N in (2.6) evaluated at any $z = (z_f)_{f \in F} \in Z_v$ is surjective; indeed, write $z_f = x_f + \mathrm{i}\, y_f$ for every $f \in F$ and suppose by contradiction that μ_N is not surjective. Then there exists $X \in \mathfrak{n}$, $X \neq 0$, such that $\sum_{f \in F} X_f\,(x_f\,x'_f + y_f\,y'_f) = 0$ for every z, $z' \in \mathbb{C}^F$. Because $z \in Z_v$, $z_{f'} \neq 0$ for every $f' \in F \setminus F_v$. By taking $f' \in F \setminus F_v$, and $z'_f = x'_f + \mathrm{i}\,y'_f = 0$ for every $f \neq f'$, as well as $z'_{f'}$ arbitrary, we conclude that $X_{f'} = 0$. Because f' is arbitrary, $X \in \iota_v(\mathbb{R}^{F_v})$. On the other hand by Lemma 3·1, $\mathfrak{n} \cap \iota_v(\mathbb{R}^{F_v}) = \{0\}$, so $X = 0$, a contradiction.

These facts lead us to the following conclusions, where in order to make the presentation self-contained, we do not assume that Z is a smooth submanifold of $\mathbb{C}^F$ of codimension equal to the dimension of N.

Proposition 3·4. *Let v be a vertex of Δ. The open subset $Z_v := {\rho_v}^{-1}(U_v) \cap Z$ of Z is a connected smooth submanifold of $\mathbb{C}^F$ of real dimension $2n+(d-n)$, where $d = \#(F)$ and $d-n = \dim N$. The action of the torus N on Z_v is free, and the projection $\rho_v : Z_v \to U_v$ exhibits Z_v as a principal N-bundle over U_v. It follows that we have a reduced phase space $M_v := Z_v/N$, which is a connected smooth symplectic $2n$-dimensional manifold, which carries an effective Hamiltonian T-action with momentum mapping as in (2.7), with Z replaced by Z_v.*

There is a unique global section $s_v : U_v \to Z_v$ of $\rho_v : Z_v \to U_v$ such that $s_v(z)_{f'} \in \mathbb{R}_{>0}$ for every $z \in U_v$ and $f' \in F \setminus F_v$. Actually, $s_v(z)_{f'} = r_{f'}(\mu_v(z))$ when $z \in U_v$ and $f' \in F \setminus F_v$, and therefore the section s_v is smooth. If $p_v : Z_v \to M_v = Z_v/N$ denotes the canonical projection, then $\psi_v := p_v \circ s_v$ is a T-equivariant symplectomorphism from U_v onto M_v, where T acts on U_v via $\mathbb{R}^{F_v}/\mathbb{Z}^{F_v}$, as in Lemma 3·3.

Remark 3·5 When z belongs to the closure $\rho_v(Z) = \overline{U_v}$ of U_v in $\mathbb{C}^{F_v}$, see Lemma 3·3, we can define $s_v(z) \in \mathbb{C}^F$ by $s_v(z)_f = z_f$ when $f \in F_v$ and $s_v(z)_{f'} = r_{f'}(\mu_v(z))$ when $f' \in F \setminus F_v$. This defines a continuous extension $s_v : \overline{U_v} \to \mathbb{C}^F$ of the mapping $s_v : U_v \to Z$. Therefore $s_v(\overline{U_v}) \subset Z$, and $\psi_v := p \circ s_v : \overline{U_v} \to M$ is a continuous extension of the diffeomorphism $\psi_v : U_v \to M_v$.

The continuous mapping $\psi_v : \overline{U_v} \to M$ is surjective, but the restriction of it to the boundary $\partial U_v := \overline{U_v} \setminus U_v$ of U_v in $\mathbb{C}^{F_v}$ is not injective. If $z^v \in \partial U_v$, then the set G of all $f' \in F \setminus F_v$ such that $s_v(z^v)_{f'} = 0$, or equivalently $\mu_T(\psi_v(z^v)) \in f'$, is not empty. The fiber of ψ_v over $\psi_v(z^v)$ is equal to the set of all $t^v \cdot z^v$, where the $t^v \in \mathbb{R}^{F_v}/\mathbb{Z}^{F_v}$ are of the form

$$t^v_f = -\sum_{g \in G} (v)^f_g\, t_g, \quad f \in F_v,$$

where $t_g \in \mathbb{R}/\mathbb{Z}$. It follows that each fiber is an orbit of some subtorus of $\mathbb{R}^{F_v}/\mathbb{Z}^{F_v}$ acting on $\mathbb{C}^{F_v}$. ⊘

Recall the definition (3.5) of the open subset Δ_v of the Delzant polytope Δ. Because the union over all vertices v of the Δ_v is equal to Δ, we have the following corollary.

Corollary 3·6. *The sets Z_v, $v \in V$, form a covering of Z. As a consequence, we recover the results mentioned in Section 2 that Z is a smooth submanifold of $\mathbb{C}^F$ of real dimension $n+d$, the action of the torus N on Z is free, and we have a reduced phase space $M := Z/N$, which is a compact and connected smooth $2n$-dimensional symplectic manifold, which carries an effective Hamiltonian T-action with momentum mapping $\mu_T : M \to T$ as in (2.7). Since Z is the level set of μ_N for the level λ_N, it follows that λ_N is a regular value of μ_N.*

Moreover, the sets M_v, $v \in V$, form an open covering of M and the $\varphi_v := (\psi_v)^{-1} : M_v \to U_v$ form an atlas of T-equivariant symplectic coordinatizations of the Hamiltonian T-space M. For each $v \in V$, we have $M_v = {\mu_T}^{-1}(\Delta_v)$, and $\mu_T|_{M_v} = \mu_v \circ \varphi_v$.

For a characterization of M_v in terms of the orbit type stratification in M for the T-action, see Corollary 5·6, which also implies that M_v is an open cell in M.

COROLLARY 3·7. *For every $f \in F$ the set ${\mu_T}^{-1}(f)$ is a real codimension two smooth compact connected smooth symplectic submanifold of M.*

For each $v \in V$, the set M_v is dense in M, and the diffeomorphism $\psi_v : U_v \to M_v$ is maximal among all diffeomorphisms from open subsets of $\mathbb{C}^{F_v}$ onto open subsets of M.

Proof. If $f \in F$, then for each $v \in V$ we have that

$$ {\mu_v}^{-1}(f) = \{z \in U_v \mid z_f = 0\} \tag{3.8} $$

if $v \in f$, that is, $f \in \Delta_v$. This follows from (3.2) and i) in the description of Δ in the beginning of Section 2. On the other hand, ${\mu_v}^{-1}(f) = \emptyset$ if $f \notin \Delta_v$. Because ${\mu_T}^{-1}(f) \cap M_v = \psi_v({\mu_v}^{-1}(f))$, and the M_v, $v \in V$, form an open covering of M, this proves the first statement. The second statement follows from the first one, because the complement of M_v in M is equal to the union of the sets ${\mu_T}^{-1}(f')$ with $f' \in F \setminus F_v$. □

Remark 3·8 It follows from the proof of Corollary 3·7, that ${\mu_T}^{-1}(f)$ is a connected component of the fixed point set in M of the of the circle subgroup $\exp(\mathbb{R}\, X_f)$ of T.

Actually, ${\mu_T}^{-1}(f)$ is a Delzant space for the action of the $(n-1)$-dimensional torus

$$ T/\exp(\mathbb{R}\, X_f), $$

with Delzant polytope $P \subset (\mathfrak{t}/(\mathbb{R}\, X_f))^*$ such that the image of P in $\mathfrak{t}^*$ under the embedding $(\mathfrak{t}/(\mathbb{R}\, X_f))^* \to \mathfrak{t}^*$ is equal to a translate of f.

In a similar way, if g is a k-dimensional face of Δ, then ${\mu_T}^{-1}(g)$ is a $2k$-dimensional Delzant space for the quotient of T by the subtorus of T which acts trivially on ${\mu_T}^{-1}(g)$. ⊘

Remark 3·9 Let $\iota : T \to \mathbb{R}^n/\mathbb{Z}^n$ be an isomorphism of tori, which allows us to let $t \in T$ act on $\mathbb{C}^n$ via $\mathbb{R}^n/\mathbb{Z}^n$ by means of

$$ (t \cdot z)_j = \mathrm{e}^{2\pi\,\mathrm{i}\,\iota(t)_j}\, z_j, \quad 1 \le j \le n. $$

Let U be a connected T-invariant open neighborhood of 0 in $\mathbb{C}^n$, provided with the symplectic form (2.4) with F replaced by $\{1, \ldots, n\}$. Let $\psi : U \to M$ be a T-equivariant symplectomorphism from U onto an open subset $\psi(U)$ of M. Because 0 is the unique fixed point for the T-action in U, and the fixed points for the T-action in M are the pre-images under μ_T of the vertices of Δ, there is a unique $v \in V$ such that $\mu_T(\psi(0)) = v$. Let $I_v : \mathbb{C}^{F_v} \to \mathbb{C}^n$ denote the complex linear extension of the tangent map of the torus isomorphism $\iota \circ (\pi \circ \iota_v)$. In terms of the notation of Lemma 3·3 and Proposition 3·4, we have that $U \subset I_v(U_v)$ and $\psi_v = \psi \circ I_v$ on ${I_v}^{-1}(U)$, which leads to an identification of ψ with the restriction of ψ_v to the connected open subset ${I_v}^{-1}(U)$ of U_v, via the isomorphism ${I_v}^{-1}$.

The ψ's, with U equal to a ball in $\mathbb{C}^n$ centered at the origin, are the equivariant symplectic ball embeddings in Pelayo [**11**], and the second statement in Corollary 3·7 shows that the diffeomorphisms ψ_v are the maximal extensions of these equivariant symplectic ball embeddings. ⊘

4. *The coordinate transformations*

Recall the description in Lemma 3·3, for every vertex v, of the open subset $U_v = \mu_v^{-1}(\Delta_v)$ of $\mathbb{C}^{F_v}$.

Let $v,\, w \in V$. Then

$$\begin{aligned} U_{v,\,w} &:= \varphi_v(M_v \cap M_w) = U_v \cap {\psi_v}^{-1} \circ \psi_w(U_w) \\ &= \{z^v \in U_v \mid (z^v)_f \neq 0 \quad \text{for every} \quad f \in F_v \setminus F_w\}. \end{aligned} \tag{4.1}$$

In this section we will give an explicit formula for the coordinate transformations

$$\varphi_w \circ {\varphi_v}^{-1} = {\psi_w}^{-1} \circ \psi_v : U_{v,\,w} \to U_{w,\,v},$$

which then leads to a description of the Delzant space M as obtained by gluing together the subsets U_v with the coordinate transformations as the gluing maps.

Let $f \in F$. Because the X_g, $g \in F_w$, form a $\mathbb{Z}$-basis of $\mathfrak{t}_{\mathbb{Z}}$, and $X_f \in \mathfrak{t}_{\mathbb{Z}}$, there exist unique integers $(w)_f^g$, $g \in F_w$, such that

$$X_f = \sum_{g \in F_w} (w)_f^g\, X_g. \tag{4.2}$$

Note that if $f \in F_w$, then $(w)_f^g = 1$ when $g = f$ and $(w)_f^g = 0$ otherwise. For the following lemma recall that r_g is defined by expression (3.7).

LEMMA 4·1. *Let $v,\, w \in V$, $z^v \in U_{v,\,w}$. Then $z^w := \varphi_w \circ {\varphi_v}^{-1}(z^v) \in U_w \subset \mathbb{C}^{F_w}$ is given by*

$$z_g^w = \prod_{f \in F_v} (z_f^v)^{(w)_f^g} \Big/ \prod_{f \in F_v \setminus F_w} |z_f^v|^{(w)_f^g} \tag{4.3}$$

if $g \in F_w \cap F_v$, and

$$z_g^w = \prod_{f \in F_v} (z_f^v)^{(w)_f^g}\, r_g(\mu_v(z^v)) \Big/ \prod_{f \in F_v \setminus F_w} |z_f^v|^{(w)_f^g} \tag{4.4}$$

if $g \in F_w \setminus F_v$.

Proof. The element $z^w \in U_w$ is determined by the condition that $s_w(z^w)$ belongs to the N-orbit of $s_v(z^v)$. That is,

$$s_w(z^w)_f = \mathrm{e}^{\mathrm{i}\, t_f}\; s_v(z^v)_f \quad \text{for every} \quad f \in F$$

for some $t \in \mathbb{R}^F$ such that

$$\sum_{f \in F} t_f\, X_f = 0. \tag{4.5}$$

It follows from (4.5), (4.2) and the linear independence of the X_g, $g \in F_w$, that $t \in \mathfrak{n}$ if and only if

$$t_g = -\sum_{f \in F \setminus F_w} (w)_f^g\, t_f \quad \text{for every} \quad g \in F_w. \tag{4.6}$$

Note that $\mu_v(z^v) = \mu_T(m) = \mu_w(z^w)$, where $m = \psi_v(z^v) = \psi_w(z^w)$. It follows from the definition of the sections s_v and s_w, see Proposition 3·4, that

i) $s_v(z^v)_f = z^v_f$ and $s_w(z^w)_f = z^w_f$ if $f \in F_v \cap F_w$,
ii) $s_v(z^v)_f = z^v_f$ and $s_w(z^w)_f = r_f(\mu_w(z^w)) = r_f(\mu_v(z^v))$ if $f \in F_v \setminus F_w$,
iii) $s_v(z^v)_f = r_f(\mu_v(z^v))$ and $s_w(z^w)_f = z^w_f$ if $f \in F_w \setminus F_v$, and
iv) $s_v(z^v)_f = r_f(\mu_v(z^v)) = r_f(\mu_w(z^w)) = s_w(z^w)_f$ if $f \in F \setminus (F_v \cup F_w)$.

It follows from ii) and iv) that $t_f = -\arg z^v_f$ and $t_f = 0$ modulo 2π if $f \in F_v \setminus F_w$ and $f \in F \setminus (F_v \cup F_w)$, respectively. Then (4.6) implies that, modulo 2π,

$$t_g = \sum_{f \in F_v \setminus F_w} (w)^g_f \arg z^v_f \quad \text{for every} \quad g \in F_w.$$

It now follows from i) and iii) that if $g \in F_w$, then $z^w_g = s_w(z^w)_g = \mathrm{e}^{\mathrm{i}\, t_g}\, s_v(z^v)_g$ is equal to

$$\mathrm{e}^{\mathrm{i}\, t_g}\, z^v_g = \prod_{f \in F_v} (z^v_f)^{(w)^g_f} \Big/ \prod_{f \in F_v \setminus F_w} |z^v_f|^{(w)^g_f}$$

if $g \in F_v$, and equal to

$$\mathrm{e}^{\mathrm{i}\, t_g}\, |z^v_g| = \prod_{f \in F_v} (z^v_f)^{(w)^g_f}\, |z^v_g| \Big/ \prod_{f \in F_v \setminus F_w} |z^v_f|^{(w)^g_f}$$

if $g \notin F_v$, respectively. Here we have used that if $g \in F_w$, then $(w)^g_f = 1$ if $f = g$ and $(w)^g_f = 0$ if $f \in F_w$, $f \neq g$. Because $|z^v_g| = r_g(\mu_v(z^v))$ if $g \notin F_v$, see (3.6) and (3.7), this completes the proof of the lemma. □

Remark 4·2 Note that $z^v \in U_{v,\,w}$ means that $z^v \in U_v$ and $z^v_f \neq 0$ if $f \in F_v \setminus F_w$. Furthermore, $z^v \in U_v$ implies that if $g \notin F_v$, then $\mu_v(z^v) \notin g$, and therefore r_g is smooth on a neighborhood of $\mu_v(z^v)$. Finally, note that if $g \in F_w$ and $f \in F_v \cap F_w$, then $(w)^g_f \in \{0,\, 1\}$, and therefore each of the factors in the right hand sides of (4.3) and (4.4) is smooth on $U_{v,\,w}$. ⊘

Remark 4·3 In (4.3) and (4.4) only the integers $(w)^g_f$ appear with $f \in F_v$ and $g \in F_w$. Let $(w\,v)$ denote the matrix $(w)^g_f$, where $f \in F_v$ and $g \in F_w$. Then $(w\,v)$ is invertible, with inverse equal to the integral matrix $(v\,w)$. These integral matrices also satisfy the cocycle condition that $(w\,v)\,(v\,u) = (w\,u)$, if $u,\, v,\, w \in V$. These properties follow from the fact that (4.2) shows that $(w\,v)$ is the matrix which maps the $\mathbb{Z}$-basis X_g, $g \in F_w$, onto the $\mathbb{Z}$-basis X_f, $f \in F_v$, of $\mathfrak{t}_{\mathbb{Z}}$. It is no surprise that these *base changes* enter in the formulas which relate the models in the vector spaces $\mathbb{C}^{F_v}$ for the different choices of $v \in V$. ⊘

Corollary 4·4. *Let, for each $v \in V$, the mapping $\mu_v : \mathbb{C}^{F_v} \to \mathfrak{t}^*$ be defined by (3.2), which is a momentum mapping for a Hamiltonian T-action via $\mathbb{R}^{F_v}/\mathbb{Z}^{F_v}$ on the symplectic vector space $\mathbb{C}^{F_v}$ as in Lemma 3·3. Define $U_v := {\mu_v}^{-1}(\Delta_v)$. If also $w \in V$, define $U_{v,\,w}$ as the right hand side of (4.1), and, if $z^v \in U_{v,\,w}$, define $\varphi_{w,\,v}(z^v) := z^w$, where $z^w \in \mathbb{C}^{F_w}$ is given by (4.3) and (4.4).*

Then $\varphi_{w,\,v}$ is a T-equivariant symplectomorphism from $U_{v,\,w}$ onto $U_{w,\,v}$ such that $\mu_w = \mu_v \circ \varphi_{v,\,w}$ on $U_{w,\,v}$. The $\varphi_{w,\,v}$ satisfy the cocycle condition $\varphi_{w,\,v} \circ \varphi_{v,\,u} = \varphi_{w,\,u}$ where the left hand side is defined. Glueing together the Hamiltonian T-spaces U_v, $v \in V$, with the momentum maps μ_v, by means of the gluing maps $\varphi_{w,\,v}$, $v,\, w \in V$, we obtain a compact *connected smooth symplectic manifold* $\widetilde{M}$ *with an effective Hamiltonian* T*-action with a common momentum map* $\widetilde{\mu} : \widetilde{M} \to T$ *such that* $\widetilde{\mu}(\widetilde{M}) = \Delta$. *In other words,* $\widetilde{M}$ *is a Delzant space for the Delzant polytope* Δ.

The Delzant space $\widetilde{M}$ is obviously isomorphic to the Delzant space $M = \mu^{-1}(\{\lambda\})/N$ introduced in Section 2, and actually the isomorphism is used in the proof that $\widetilde{M}$ is a Delzant space for the Delzant polytope Δ. The only purpose of Corollary 4·4 is to exhibit the Delzant space as obtained from gluing together the U_v, $v \in V$, by means of the gluing maps $\varphi_{v,w}$, v, $w \in V$.

5. *The toric variety*

Let $\mathbb{T} := \{z \in \mathbb{C} \mid |z| = 1\}$ denote the unit circle in the complex plane. The mapping $t \mapsto u$ where $u_f = \mathrm{e}^{2\pi \, \mathrm{i} \, t_f}$ for every $f \in F$ is an isomorphism from the torus $\mathbb{R}^F/\mathbb{Z}^F$ onto $\mathbb{T}^F$, where $\mathbb{T}^F$ acts on $\mathbb{C}^F$ by means of coordinatewise multiplication and $\mathbb{R}^F/\mathbb{Z}^F$ acted on $\mathbb{C}^F$ via the isomorphism from $\mathbb{R}^F/\mathbb{Z}^F$ onto $\mathbb{T}^F$. The complexification $\mathbb{T}_{\mathbb{C}}$ of the compact Lie group $\mathbb{T}$ is the multiplicative group $\mathbb{C}^\times$ of all nonzero complex numbers, and the complexification of $\mathbb{T}^F$ is equal to $\mathbb{T}_{\mathbb{C}}^F := (\mathbb{T}_{\mathbb{C}})^F = (\mathbb{C}^\times)^F$, which also acts on $\mathbb{C}^F$ by means of coordinatewise multiplication.

The complexification $N_{\mathbb{C}}$ of N is the subgroup $\exp(\mathfrak{n}_{\mathbb{C}})$ of $U_{\mathbb{C}}^F$, where $\mathfrak{n}_{\mathbb{C}} := \mathfrak{n} \oplus \mathrm{i}\, \mathfrak{n} \subset \mathbb{C}^F$ denotes the complexification of $\mathfrak{n}$, viewed as a complex linear subspace, a complex Lie subalgebra, of the Lie algebra $\mathbb{C}^F$ of $\mathbb{T}_{\mathbb{C}}^F$. In view of (4.6), we have, for every $v \in V$, that $N_{\mathbb{C}}$ is equal to the set of all $t \in \mathbb{T}_{\mathbb{C}}^F$ such that

$$t_g = \prod_{f \in F \setminus F_v} t_f{}^{-(v)^g_f}, \quad g \in F_v. \tag{5.1}$$

This implies that $N_{\mathbb{C}}$ is a closed subgroup of $\mathbb{T}_{\mathbb{C}}^F$ isomorphic to $\mathbb{T}_{\mathbb{C}}^{F \setminus F_v}$, and therefore $N_{\mathbb{C}}$ is a reductive complex algebraic group.

If we define

$$\mathbb{C}_v^F := \{z \in \mathbb{C}^F \mid z_f \neq 0 \quad \text{for every} \quad f \in F \setminus F_v\}, \tag{5.2}$$

then it follows from (5.1) that the action of $N_{\mathbb{C}}$ on $\mathbb{C}_v^F$ is free and proper. It follows that the action of $N_{\mathbb{C}}$ on

$$\mathbb{C}_V^F = \bigcup_{v \in V} \mathbb{C}_v^F \tag{5.3}$$

is free.

LEMMA 5·1. *The action of* $N_{\mathbb{C}}$ *on* $\mathbb{C}_V^F$ *is proper.*

Proof. As the referee observed, this does not follow immediately from the properness of the $N_{\mathbb{C}}$-action on each of the $\mathbb{C}_v^F$'s, and because we did not find a proof in the literature, we present one here. Also, G. Schwarz observed that in view of Luna's slice theorem it is sufficient to prove that the $N_{\mathbb{C}}$-orbits are closed subsets of $\mathbb{C}_V^F$ but we could not find a proof for the closedness of the orbits which is much simpler than the proof of the properness of the action. Finally, the statement of the lemma is implicitly contained in the statement in Audin [**2**, bottom of p. 155] that $\mathcal{U}_\Sigma \to X_\Sigma$ is a principal $K_{\mathbb{C}}$-bundle.

For each subset E of F, let $C(E) = \sum_{f \in E} \mathbb{R}_{\geq 0} \, X_f$ denote the polyhedral cone in $\mathfrak{t}$ spanned by the vectors X_f, $f \in E$. Let v, $w \in V$, $X \in C(F_v) \cap C(F_w)$, and $\xi \in \Delta$. That is,

$$X = \sum_{f \in F_v} c_f \, X_f = \sum_{g \in F_w} d_g \, X_g,$$

with $c_f, d_g \in \mathbb{R}_{\geq 0}$, and $\langle X_f, \, \xi \rangle + \lambda_f \geq 0$ for every $f \in F$. Because $\langle X_f, \, v \rangle + \lambda_f = 0$ for every $f \in F_v$, it follows that $\langle X, \, \xi - v \rangle \geq 0$, with equality if and only if $c_f = 0$ or $\xi \in f$ for every $f \in F_v$.

Similarly $\langle X, \, \xi - w \rangle \geq 0$, with equality if and only if $d_g = 0$ or $\xi \in g$ for every $g \in F_w$. For $\xi = (1/2)\,(v + w) \in \Delta$ we have $f \in F_v \cap F_w$ if $\xi \in f \in F_v$ or $\xi \in g \in F_w$, hence $c_f = 0$ for every $f \in F_v \setminus F_w$ and $d_g = 0$ for every $g \in F_w \setminus F_v$. We therefore have proved that the collection of simplicial cones $C(F_v)$, $v \in V$, has the *fan property* of Demazure [**4**, Déf. 1 in §4] that $C(F_v) \cap C(F_w) = C(F_v \cap F_w)$ for every $v, \, w \in V$.

The argument below that the fan property implies the properness of the $N_{\mathbb{C}}$-action on $\mathbb{C}^F_V$ is inspired by the proof of Danilov [**5**, bottom of p. 133] that a toric variety defined by a fan is separated.

What we have to prove is that if $x \in \mathbb{C}^F_V$ is close to $x^0 \in \mathbb{C}^F_V$, $t \in N_{\mathbb{C}}$, and $y = t \cdot x \in \mathbb{C}^F_V$ is close to $y^0 \in \mathbb{C}^F_V$, then t remains in a compact subset of $N_{\mathbb{C}}$, that is, t_f remains bounded and bounded away from 0 for every $f \in F$. It follows from (5.3) that we have $x^0 \in \mathbb{C}^F_v$ and $y^0 \in \mathbb{C}^F_w$ for some $v, \, w \in V$. Then (5.2) implies that for every $f \in F \setminus (F_v \cup F_w)$ both x_f and y_f remain bounded and bounded away from zero, hence $t_f = y_f / x_f$ remains bounded and bounded away from zero. The fan property implies that there exists a linear form ξ on $\mathfrak{t}$ such that $\langle X_f, \, \xi \rangle > 0$ for every $f \in F_v \setminus F_w$, $\langle X_f, \, \xi \rangle = 0$ for every $f \in F_v \cap F_w$, and $\langle X_g, \, \xi \rangle < 0$ for every $g \in F_w \setminus F_v$. We can arrange that $\langle X_f, \, \xi \rangle \in \mathbb{Z}$ for every $f \in F_v$, which implies that $\langle X_f, \, \xi \rangle \in \mathbb{Z}$ for every $f \in F$ because the X_f, $f \in F_v$ form a $\mathbb{Z}$-basis of F.

For each $f \in F_v \setminus F_w$ it follows from (5.1) that

$$t_f = \prod_{g \in F_w \setminus F_v} t_g^{-(v)^f_g} \prod_{h \in F \setminus (F_v \cup F_w)} t_h^{-(v)^f_h},$$

where the second factor remains bounded and bounded away from zero. Using (4.2) we therefore obtain

$$\prod_{f \in F_v \setminus F_w} t_f^{\langle X_f, \, \xi \rangle} = \prod_{g \in F_w \setminus F_v} t_g^{-\langle X_g, \, \xi \rangle} \, \varphi, \tag{5.4}$$

where the factor φ remains bounded and bounded away from zero. It follows from $y^0 \in \mathbb{C}^F_w$ that y_f remains bounded away from zero for every $f \in F_v \setminus F_w$, and because x_f remains bounded, $t_f = y_f / x_f$ remains bounded away from zero. On the other hand it follows from $x^0 \in F_v$ that, for each $g \in F_w \setminus F_v$, x_g remains bounded away from zero, and because y_g remains bounded, it follows that $t_g = y_g / x_g$ remains bounded. Because $\langle X_g, \, \xi \rangle < 0$ for every $g \in F_w \setminus F_v$, it follows that the right hand side in (5.4) remains bounded, and therefore the left hand side as well. Because $\langle X_f, \, \xi \rangle > 0$ for every $f \in F_v \setminus F_w$, and each t_f, $f \in F_v \setminus F_w$, remains bounded away from zero, it follows that t_f remains bounded for every $f \in F_v \setminus F_w$. This in turn implies that, for each $f \in F_v \setminus F_w$, $x_f = y_f / t_f$ remains bounded away from zero, because y_f does so. Therefore $x^0 \in \mathbb{C}^F_w$, and because also $y^0 \in \mathbb{C}^F_w$, it follows that the t remain in a compact subset of $N_{\mathbb{C}}$ because the $N_{\mathbb{C}}$-action on $\mathbb{C}^F_w$ is proper. $\square$

Therefore the $N_{\mathbb{C}}$-orbit space

$$M^{\mathrm{toric}} := \mathbb{C}^F_V / N_{\mathbb{C}} \tag{5.5}$$

has the unique structure of a Hausdorff complex analytic manifold of complex dimension n such that the canonical projection from $\mathbb{C}^F_V$ onto M^{toric} exhibits $\mathbb{C}^F_V$ as a principal $N_{\mathbb{C}}$-bundle over M^{toric}. On M^{toric} we still have the complex analytic action of the complex Lie group group $\mathbb{T}^F_{\mathbb{C}}/N_{\mathbb{C}}$, which is isomorphic to the complexification $T_{\mathbb{C}}$ of our real torus T induced by the projection π. The complex analytic manifold M^{toric} together with the complex analytic action of $T_{\mathbb{C}}$ on it is the *toric variety defined by the polytope* Δ in the title of this section.

If $v \in V$ and $z \in \mathbb{C}^F_v$, then it follows from (5.1) that there is a unique $t \in N_{\mathbb{C}}$ such that $t_f = z_f$ for every $f \in F \setminus F_v$, or in other words, $z = t \cdot \zeta$, where $\zeta \in \mathbb{C}^F$ is such that $\zeta_f = 1$ for every $f \in F \setminus F_v$. Let $S_v : \mathbb{C}^{F_v} \to \mathbb{C}^F_v$ be defined by $S_v(z^v)_f = z^v_f$ when $f \in F$ and $S_v(z^v) = 1$ when $f \in F \setminus F_v$, as in Audin [**2**, p. 159]. If $P_v : \mathbb{C}^F_v \to \mathbb{C}^F_v/N_{\mathbb{C}}$ denotes the canonical projection from $\mathbb{C}^F_v$ onto the open subset $M^{\mathrm{toric}}_v := \mathbb{C}^F_v/N_{\mathbb{C}}$ of M^{toric}, then $\Psi_v := P_v \circ S_v$ is a complex analytic diffeomorphism from $\mathbb{C}^{F_v}$ onto M^{toric}_v. It is $T_{\mathbb{C}}$-equivariant if we let $T_{\mathbb{C}}$ act on $\mathbb{C}^{F_v}$ via $\mathbb{T}^{F_v}_{\mathbb{C}}$ as in Lemma 3·3. We use the diffeomorphism $\Phi_v := {\Psi_v}^{-1}$ from M^{toric}_v onto $\mathbb{C}^{F_v}$ as a coordinatization of the open subset M^{toric}_v of M^{toric}.

If $v,\, w \in V$, then

$$\begin{aligned} U^{\mathrm{toric}}_{v,\,w} &:= \Phi_v(M^{\mathrm{toric}}_v \cap M^{\mathrm{toric}}_w) = \mathbb{C}^{F_v} \cap {\Psi_v}^{-1} \circ \Psi_w(\mathbb{C}^{F_w}) \\ &= \{z^v \in \mathbb{C}^{F_v} \mid (z^v)_f \neq 0 \quad \text{for every} \quad f \in F_v \setminus F_w\}. \end{aligned} \tag{5.6}$$

Moreover, with a similar argument as for Lemma 4·1, actually much simpler, we have that for every $z^v \in U^{\mathrm{toric}}_{v,\,w}$ the element $z^w := \Phi_w \circ {\Phi_v}^{-1}(z^v) \in \mathbb{C}^{F_w}$ is given by

$$z^w_g = \prod_{f \in F_v} (z^v_f)^{(w)^g_f}, \quad g \in F_w, \tag{5.7}$$

where we define $(z^v_f)^0 = 1$ when $z^v_f = 0$, which can happen when $f \in F_v \cap F_w$. In this way the coordinate transformation $\Phi_w \circ {\Phi_v}^{-1}$ is a Laurent monomial mapping, much simpler than the coordinate transformation (4.3), (4.4). It follows that the toric variety M^{toric} can be alternatively described as obtained by gluing the n-dimensional complex vector spaces $\mathbb{C}^{F_v}$, $v \in V$, together, with the maps (5.7) as the gluing maps. This is the kind of toric varieties as introduced by Demazure [**4**, Sec. 4].

For later use we mention the following observation of Danilov [**5**, Th. 9.1], which is also of interest in itself.

LEMMA 5·2. *M^{toric} is simply connected.*

Proof. Let $w \in V$. It follows from (5.6), for all $v \in V$, that the complement of M^{toric}_w in M_{toric} is equal to the union of finitely closed complex analytic submanifolds of complex codimension one, whereas M^{toric}_w is contractible because it is diffeomorphic to the complex vector space $\mathbb{C}^{F_w}$. Because complex codimension one is real codimension two, any loop in M^{toric} with base point in M^{toric}_w can be slightly deformed to such a loop which avoids the complement of M^{toric}_w in M_{toric}, that is, which is contained in M^{toric}_w, after which it can be contracted within M^{toric}_w to the base point in M^{toric}_w. $\square$

Recall the definition in Section 2 of the reduced phase space $M = Z/N$.

THEOREM 5·3. *The identity mapping from Z into $\mathbb{C}_V^F$, followed by the canonical projection P from $\mathbb{C}_V^F$ to $M^{\rm toric} = \mathbb{C}_V^F/N_{\mathbb{C}}$, induces a T-equivariant diffeomorphism ϖ from $M = Z/N$ onto $M^{\rm toric}$. It follows that each $N_{\mathbb{C}}$-orbit in $\mathbb{C}_V^F$ intersects Z in an N-orbit in Z.*

Proof. Because N is a closed Lie subgroup of $N_{\mathbb{C}}$, we have that the mapping $P : Z \to \mathbb{C}_V^F/N_{\mathbb{C}}$ induces a mapping $\varpi : Z/N \to \mathbb{C}_V^F/N_{\mathbb{C}}$, which moreover is smooth.

If $v \in V$, $z \in Z_v$, then it follows from (5.1) that the t_f, $f \in F \setminus F_v$, of an element $t \in N_{\mathbb{C}}$ can take arbitrary values, and therefore the $|z_f|$, $f \in F \setminus F_v$ can be moved arbitrarily by means of infinitesimal $N_{\mathbb{C}}$-actions. Because Z is defined by prescribing the $|z_f|$, $f \in F \setminus F_v$, as a smooth function of the z_f, $f \in F_v$, and the Z_v, $v \in V$, form an open covering of Z, this shows that at each point of Z the $N_{\mathbb{C}}$-orbit is transversal to Z, which implies that ϖ is a submersion.

It follows that $\varpi(M)$ is an open subset of $M^{\rm toric}$. Because M is compact and ϖ is continuous, $\varpi(M)$ is compact, and therefore a closed subset of $M^{\rm toric}$. Because $M^{\rm toric}$ is connected, the conclusion is that $\varpi(M) = M^{\rm toric}$, that is, ϖ is surjective.

Because ϖ is a surjective submersion, $\dim_{\mathbb{R}} M = 2n = \dim_{\mathbb{R}} M^{\rm toric}$, and M is connected, we conclude that ϖ is a covering map. Because $M^{\rm toric}$ is simply connected, see Lemma 5·2, we conclude that ϖ is injective, that is, ϖ is a diffeomorphism. □

Remark 5·4 Theorem 5·3 is the last statement in Delzant [**3**], with no further details of the proof. Audin [**2**, Prop. 3.1.1] gave a proof using gradient flows, whereas the injectivity has been proved in [**8**, Sec. A1.2] using the principle that the gradient of a strictly convex function defines an injective mapping. ⊘

Note that in the definition of the toric variety $M^{\rm toric}$, the real numbers λ_f, $f \in F$, did not enter, whereas these numbers certainly enter in the definition of M, the symplectic form on M, and the diffeomorphism $\varpi : M \to M^{\rm toric}$. Therefore the symplectic form $\sigma_\lambda^{\rm toric} := (\varpi^{-1})^*(\sigma)$ on $M^{\rm toric}$ will depend on the choice of $\lambda \in \mathbb{R}^F$. On the symplectic manifold $(M^{\rm toric}, \sigma_\lambda^{\rm toric})$, the action of the maximal compact subgroup T of $T_{\mathbb{C}}$ is Hamiltonian, with momentum mapping equal to

$$\mu_\lambda^{\rm toric} := \mu \circ \varpi^{-1} : M^{\rm toric} \to \mathfrak{t}^*, \tag{5.8}$$

where $\mu_\lambda^{\rm toric}(M^{\rm toric}) = \Delta$, where we note that Δ in (2.1) depends on λ.

In the following lemma we compare the reduced phase space coordinatizations with the toric variety coordinatizations.

LEMMA 5·5. *Let $v \in V$. Then $M_v^{\rm toric} = \varpi(M_v)$, and*

$$\theta_v := {\Psi_v}^{-1} \circ \varpi \circ \psi_v \tag{5.9}$$

is a $\mathbb{T}^{F_v}$-equivariant diffeomorphism from U_v onto $\mathbb{C}^{F_v}$.

For each $z^v \in U_v$, the element $\zeta^v := \theta_v(z^v)$ is given in terms of z^v by

$$\zeta_f^v = z_f^v \prod_{f' \in F \setminus F_v} r_{f'}(\mu_v(z^v))^{(v)_{f'}^f}, \quad f \in F_v, \tag{5.10}$$

where the functions $r_{f'} : \Delta \to \mathbb{R}_{\geq 0}$ are given by (3.7). We have

$$\mu_v(z_v) = \mu_T(\psi_v(z^v)) = \mu_\lambda^{\rm toric}(\Psi_v(\zeta^v)), \tag{5.11}$$

and $z^v = {\theta_v}^{-1}(\zeta_v)$ *is given in terms of* ζ^v *by*

$$z^v_f = \zeta^v_f \prod_{f' \in F \setminus F_v} r_{f'}(\xi)^{-(v)^f_{f'}}, \quad f \in F_v, \tag{5.12}$$

coordtranstoric where ξ *is the element of* Δ *equal to the right hand side of (*5.11*).*

Proof. It follows from Lemma 3·3 and the paragraph preceding Proposition 3·4 that if $z^v \in \rho^v(Z)$, then $z^v \in U_v$ if and only if $z^v_{f'} \neq 0$ for every $f' \in F \setminus F_v$. That is, the set Z_v in Proposition 3·4 is equal to $Z \cap \mathbb{C}^F_v$. It therefore follows from Theorem 5·3 that each $N_{\mathbb{C}}$-orbit in the $N_{\mathbb{C}}$-invariant subset $\mathbb{C}^F_v$ of $\mathbb{C}^F_V$ intersects the N-invariant subset Z_v of Z in an N-orbit in Z_v, that is,

$$M^{\mathrm{toric}}_v = P_v(\mathbb{C}^F_v) = \varpi(p_v(Z_v)) = \varpi(M_v).$$

If $z^v \in U_v$, then Proposition 3·4 implies that $s_v(z^v)_f = z^v_f$ for every $f \in F_v$ and

$$s_v(z^v)_{f'} = r_{f'}(\mu_v(z^v)), \quad f' \in F \setminus F_v.$$

If we define $t \in \mathbb{T}^F_{\mathbb{C}}$ by

$$t_{f'} = r_{f'}(\mu_v(z^v))^{-1}, \quad f' \in F \setminus F_v,$$
$$t_f = \prod_{f' \in F \setminus F_v} r_{f'}(\mu_v(z^v))^{(v)^f_{f'}}, \quad f \in F_v,$$

then $(t \cdot s_v(z^v))_{t'} = 1$ for every $t' \in F \setminus F_v$ and, for every $f \in F_v$, $\zeta^v_f := (t \cdot s_v)_f$ is equal to the right hand side of (5.10). That is, $t \cdot s_v(z^v) = S_v(\zeta^v)$, see the definition of S_v in the paragraph preceding (5.6). On the other hand, it follows from (5.1) that $t \in N_{\mathbb{C}}$, and therefore

$$\Psi_v(\zeta_v) = P_v(t \cdot s_v(z^v)) = P_v(s_v(z^v)) = \varpi \circ p_v(s_v(z^v)) = \varpi \circ \psi_v(z^v),$$

that is, $\zeta^v = {\Psi_v}^{-1} \circ \varpi \circ \psi_v(z^v)$. □

COROLLARY 5·6. *Let* s *be the relative interior of a face of* Δ*. Then* ${\mu_T}^{-1}(s)$ *is equal to a stratum* S *of the orbit type stratification in* M *of the* T*-action, and also equal to the preimage under* $\varpi : M \to M^{\mathrm{toric}}$ *of a* $T_{\mathbb{C}}$*-orbit in* M^{toric}*. If* $s = \{v\}$ *for a vertex* v*, then* ${\mu_T}^{-1}(s) = \{m_v\}$ *for the unique fixed point* m_v *in* M *for the* T*-action such that* $\mu_T(m_v) = v$*.*

The mapping $s \mapsto {\mu_T}^{-1}(s)$ *is a bijection from the set* Σ_Δ *of all relative interiors of faces of* Δ *onto the set* Σ *of all strata of the orbit type stratificiation in* M *for the action of* T*. If* s, $s' \in \Sigma_\Delta$ *then* s *is contained in the closure of* s' *in* Δ *if and only if* ${\mu_T}^{-1}(s)$ *is contained in the closure of* ${\mu_T}^{-1}(s')$ *in* M*.*

The domain of definition M_v *of* φ_v *in* M *is equal to the union of the* $S \in \Sigma$ *such that* m_v *belongs to the closure of* S *in* M*. The domain of definition* $M^{\mathrm{toric}}_v = \varpi(M_v)$ *of* Φ_v *is equal to the union of the corresponding strata of the* T*-action in* M^{toric}*, each of which is a single* $T_{\mathbb{C}}$*-orbit in* M^{toric}*.* M_v *and* M^{toric}_v *are open cells in* M *and* M^{toric}*, respectively.*

Proof. There exists a vertex v of Δ such that v belongs to the closure of s in $\mathfrak{t}^*$, which implies that s is disjoint from all $f' \in F \setminus F_v$. Let $F_{v,\,s}$ denote the set of all $f \in F_v$ such that $s \subset f$, where $F_{v,\,s} = \emptyset$ if and only if s is the interior of Δ. For any subset G of F_v, let $\mathbb{C}^{F_v}_G$ denote the set of all $z \in \mathbb{C}^{F_v}$ such that $z_f = 0$ if $f \in G$ and $z_f \neq 0$ if $f \in F_v \setminus G$. It follows from $\mu_v = \mu_T \circ \psi_v$ and (3.8) that $\psi_v^{-1}({\mu_T}^{-1}(s))$ is equal to $U_v \cap \mathbb{C}^{F_v}_G$ with

$G = F_{v,\,s}$. The diffeomorphism θ_v maps this set onto the set $\mathbb{C}^{F_v}_G$ with $G = F_{v,\,s}$. Because the sets of the form $\mathbb{C}^{F_v}_G$ with $G \subset F_v$ are the strata of the orbit type stratification of the $\mathbb{T}^{F_v}$-action on $\mathbb{C}^{F_v}$, and also equal to the $(\mathbb{T}_{\mathbb{C}})^{F_v}$-orbits in $\mathbb{C}^{F_v}$, the first statement of the corollary follows.

The second statement follows from ${\mu_v}^{-1}(\{v\}) = \{0\}$ and the fact that 0 is the unique fixed point of the $\mathbb{T}^{F_v}$-action in U_v.

If $s \in \Sigma_\Delta$ and $v \in V$, then M_v belongs to the closure of ${\mu_T}^{-1}(s)$ if and only if s is not contained in any $f' \in F \setminus F_v$. This proves the characterization of the domain of definition $M_v := Z_v/N = {\mu_T}^{-1}(\Delta_v)$ of φ_v. The last statement follows from the fact that Φ_v is a diffeomorphism from M_v^{toric} onto the vector space $\mathbb{C}^{F_v}$, and ϖ is a diffeomorphism from M_v onto M_v^{toric}. $\square$

The stratification of M^{toric} by $T_{\mathbb{C}}$-orbits is one of the main tools in the survey of Danilov [**5**] on the geometry of toric varieties.

Remark 5·7 If $v,\, w \in V$, then

$$\varphi_w \circ {\varphi_v}^{-1} = {\psi_w}^{-1} \circ \psi_v = {\theta_w}^{-1} \circ ({\Psi_w}^{-1} \circ \Psi_v) \circ \theta_v = {\theta_w}^{-1} \circ (\Phi_w \circ {\Phi_v}^{-1}) \circ \theta_v.$$

Using the formula (5.7) for $\Phi_w \circ {\Phi_v}^{-1}$, this can be used in order to obtain the formulas (4.3), (4.4) as a consequence of (5.10). In the proof, it is used that $\xi := \mu_v(z^v) = \mu_w(z^w)$, $|z^v_f| = r_f(\xi)$ if $f \in F_v \setminus F_w$, and

$$\sum_{f \in F_v} (v)^f_{f'}\,(w)^g_f = (w)^g_{f'}$$

if $f' \in F \setminus F_v$ and $g \in F_w$. $\oslash$

In the following corollary we describe the symplectic form $\sigma_\lambda^{\mathrm{toric}}$ on the toric variety M^{toric} in the toric variety coordinates.

Corollary 5·8. *For each $v \in V$, the symplectic form $({\Phi_v}^{-1})^*(\sigma_\lambda^{\mathrm{toric}})$ on $\mathbb{C}^{F_v}$ is equal to $({\theta_v}^{-1})^*(\sigma_v)$, where σ_v is the standard symplectic form on $\mathbb{C}^{F_v}$ given by (*3.4*).*

Because $r_{f'}(\mu_v(z^v))^2$ is an inhomogeneous linear function of the quantities $|z^v_f|^2$, it follows from (5.10) that the equations which determine the $|z^v_f|^2$ in terms of the quantities $|\zeta^v_f|^2$ are n polyomial equations for the n unknowns $|z^f_v|^2$, $f \in F_v$, where the coefficients of the polyomials are inhomogeneous linear functions of the $|\zeta^v_f|$, $f \in F_v$. In this sense the $|z^v_f|^2$, $f \in F_v$, are algebraic functions of the $|\zeta^v_f|^2$, $f \in F_v$, and substituting these in (5.10) we obtain that the diffeomorphism ${\theta_v}^{-1}$ from $\mathbb{C}^{F_v}$ onto U_v is an algebraic mapping. If Δ is a simplex, when M^{toric} is the n-dimensional complex projective space, we have an explicit formula for ${\theta_v}^{-1}$, see Subsection 7·1. However, already in the case that Δ is a planar quadrangle, when M^{toric} is a complex two-dimensional Hirzebruch surface, we do not have an explicit formula for ${\theta_v}^{-1}$. See Subsection 7·2.

Summarizing, we can say that in the toric variety coordinates the complex structure is the standard one and the coordinate transformations are the relatively simple Laurent monomial transformations (5.7). However, in the toric variety coordinates the λ-dependent symplectic form in general is given by quite complicated algebraic functions. On the other hand, in the reduced phase space coordinates the symplectic form is the standard one, but the coordinate transformations (4.3), (4.4) are more complicated. Also the complex structure in the reduced phase space coordinates, which depends on λ, is given by more complicated formulas.

Remark 5·9 It is a challenge to compare the formula in Corollary 5·8 for the symplectic form in toric variety coordinates with Guillemin's formula in [**8**, Th. 3.5 on p. 141] and [**9**, (1.3)]. Note that in the latter the pullback by means of the momentum mapping appears of a function on the interior of Δ, where in general we do not have a really explicit formula for the momentum mapping in toric variety coordinates. ⊘

6. *Cohomology classes of Kähler forms on toric varieties*

For the construction of the toric variety by gluing the $\mathbb{C}^{F_v}$, $v \in V$ together by means of the gluing maps (5.7), one only needs the integral $F_w \times F_v$-matrices $(w\,v)_f^g := (w)_f^g \in \mathbb{Z}$, $g \in F_w$, $f \in F_v$ as the data. The Laurent monomial coordinate transformations $U_{v,\,w}^{\text{toric}} \to U_{w,\,v}^{\text{toric}} : z^v \mapsto z^w$ are diffeomorphisms if and only if the integral matrices $(w\,v)$ are invertible, and the gluing defines an equivalence relation if and only if the matrices satisfy the cocycle condition $(u\,w) = (u\,v)\,(v\,w)$ for all $u,\, v,\, w \in V$. If this holds, then the same gluing procedure allows to glue the $\mathbb{R}^{F_v}$, $\mathbb{Z}^{F_v}$, $\mathbb{T}_v^F$, and $(\mathbb{C}^\times)^{F_v}$ together to an n-dimensional vector space $\mathfrak{t}$, an integral lattice $\mathfrak{t}_{\mathbb{Z}}$ in $\mathfrak{t}$, an n-dimensional torus T, and the complexification $T_{\mathbb{C}}$ of T, respectively, where T is the unique maximal compact subgroup of $T_{\mathbb{C}}$. Here the exponentation $t \mapsto \mathrm{e}^{2\pi\,\mathrm{i}\,t}$ on each coordinate defines the isomorphism $\mathfrak{t}/\mathfrak{t}_{\mathbb{Z}} \xrightarrow{\sim} T$. For each $v \in V$ and $f \in F_v$ the standard $\mathbb{Z}$-basis vector $e_f \in \mathbb{Z}^{F_v}$ is mapped to an element $X_f \in \mathfrak{t}$, where the X_f, $f \in F_v$, form a $\mathbb{Z}$-basis of $\mathfrak{t}_{\mathbb{Z}}$. The manifold M^{toric} obtained by means of the gluing process is Hausdorff = separated in the algebraic geometric terminology, if and only if the vectors $X_f \in \mathfrak{t}$, $f \in F$, have the fan property. The toric variety M^{toric} is compact if and only if the fan is *complete*, which means that $\mathfrak{t}$ is equal to the union of all the cones $C(F_v)$, $v \in V$.

In the above construction, F and V are just abstract finite sets, and in particular do not yet have the interpretation of being the set of faces and vertices, respectively, of a Delzant polytope in $\mathfrak{t}^*$. We now will construct, for each element $\lambda \in \mathbb{R}^F$ satisfying suitable conditions, a Delzant polytope $\Delta = \Delta_\lambda$, such that F and V can be identified as the set of faces and vertices of Δ_λ, respectively. For a given $\lambda \in \mathbb{R}^F$, there is a unique solution $\xi = v_\lambda \in \mathfrak{t}^*$ of the linear equations $\langle X_f,\, \xi\rangle + \lambda_f = 0$ for all $f \in F_v$. On the other hand we have the subset $\Delta = \Delta_\lambda$ of $\mathfrak{t}^*$ defined by (2.1). The condition of having a complete fan is equivalent to the existence of a choice of λ_f's such that Δ_λ is a convex polytope in $\mathfrak{t}^*$ with the v_λ, $v \in V$ as its vertices. This means that $\langle X_f,\, v_\lambda\rangle + \lambda_f > 0$ for each $f \in F \setminus F_v$. These λ's form a convex open cone Λ in $\mathbb{R}^F$. Identifying $v_\lambda \in \mathfrak{t}^*$ with v, we obtain for each $\lambda \in \Lambda$ the symplectic form $\sigma_\lambda^{\text{toric}} := ({\varpi_\lambda}^{-1})^*\sigma$ on M^{toric}, where $\varpi_\lambda = \varpi$ is the diffeomorphism from the Delzant space $M = M_\lambda = Z_\lambda/N$ onto M^{toric}. Here $Z = Z_\lambda$ denotes the level set of the momentum mapping μ_N at the level $\iota_{\mathfrak{n}}^*(\lambda)$.

Let $[\sigma_\lambda^{\text{toric}}] \in \mathrm{H}^2_{\text{de Rham}}(M^{\text{toric}})$ denote the de Rham cohomology class of $\sigma_\lambda^{\text{toric}}$. For each $f \in F$ the linear form $\epsilon_f : \mathbb{C}^F \to \mathbb{C} : z \mapsto z_f$ induces a surjective homomorphism of tori $N \to \mathbb{T}$, hence an isomorphism $N/\ker(\epsilon_f|_N) \xrightarrow{\sim} \mathbb{T}$, and therefore $N/\ker(\epsilon_f|_N)$ is a circle group. It follows from Duistermaat and Heckman [**6**, (2.10)] that, for each $f \in F$,

$$\frac{\partial[\sigma_\lambda^{\text{toric}}]}{\partial\lambda_f} = \iota_{\text{de Rham}}(c_f). \tag{6.1}$$

Here $c_f \in \mathrm{H}^2(M^{\mathrm{toric}}, \mathbb{Z})$ denotes the pullback under ${\varpi_\lambda}^{-1}$ of the Chern class of the principal $N/\ker(\epsilon_f|_N)$-bundle $Z_\lambda/(\ker \epsilon_f|_N)$ over Z_λ/N, and $\iota_{\text{de Rham}}$ denotes the canonical mapping from $\mathrm{H}^2(M^{\mathrm{toric}}, \mathbb{Z})$ to $\mathrm{H}^2(M^{\mathrm{toric}}, \mathbb{Z}) \otimes \mathbb{R} \xrightarrow{\sim} \mathrm{H}^2(M^{\mathrm{toric}}, \mathbb{R}) \xrightarrow{\sim} \mathrm{H}^2_{\text{de Rham}}(M^{\mathrm{toric}})$, where the first mapping is the canonical tensor product mapping, and the second and third arrow are the universal coefficient theorem and the de Rham isomorphism, respectively.

Let $\lambda \in \Lambda$ and $\rho \in \mathbb{R}_{>0}$. The multiplication α_ρ by $\rho^{1/2}$ is an N-equivariant diffeomorphism from Z_λ onto $Z_{\rho\,\lambda}$, and $\alpha_\rho^*\,\iota_{Z_{\rho\,\lambda}}^*\sigma_\lambda^{\mathrm{toric}} = \rho\;\iota_{Z_\lambda}^*\sigma_\lambda^{\mathrm{toric}}$. Therefore, if β_ρ denotes the diffeomorphism from Z_λ/N onto $Z_{\rho\,\lambda}/N$ induced by α_ρ, and $\gamma_\rho := \varpi_{\rho\,\lambda} \circ \beta_\rho \circ {\varpi_\lambda}^{-1}$, then ${\gamma_\rho}^*\sigma_{\rho\,\lambda}^{\mathrm{toric}} = \rho\;\sigma_\lambda^{\mathrm{toric}}$. Because the diffeomorphism γ_ρ of M^{toric} is homotopic to the identity γ_1, its action on $\mathrm{H}^2_{\text{de Rham}}(M^{\mathrm{toric}})$ is trivial, and it follows that $[\sigma_{\rho\,\lambda}^{\mathrm{toric}}] = \rho\,[\sigma_\lambda^{\mathrm{toric}}]$ for every $\rho \in \mathbb{R}_{>0}$. In combination with (6.1) this leads to the formula

$$[\sigma_\lambda^{\mathrm{toric}}] = \sum_{f \in F} \lambda_f\;\iota_{\text{de Rham}}(c_f), \tag{6.2}$$

where we have the identities (6.3) below for the Chern classes c_f, $f \in F$. The idea of using (6.1) has also been used by Delzant [**3**, p. 319] and Guillemin [**8**, Th. 2.7]. For a different proof, see Guillemin [**9**, Th. 6.3].

The mapping $Z_\lambda \to \mathbb{Z}_\lambda \times \mathbb{T} : z \mapsto (z,\,1)$ induces an isomorphism from the circle bundle $Z_\lambda/\ker(\epsilon_f|_N)$ over Z_λ/N onto the circle bundle $Z_\lambda \times_N \mathbb{T}$ over Z_λ/N, where $t \in N$ acts on $Z_\lambda \times \mathbb{T}$ by sending $(z,\,u)$ to $(t \cdot z,\,t_f\,u)$. The embedding of $Z_\lambda \times \mathbb{T}$ in $\mathbb{C}^F_V \times \mathbb{T}$ induces an isomorphism from the latter circle bundle onto the circle bundle $\mathbb{C}^F_V \times_{N_\mathbb{C}} \mathbb{T}$ over $\mathbb{C}^F_V/N_\mathbb{C} = M^{\mathrm{toric}}$, of which the Chern class is equal to the Chern class of the associated holomorphic complex line bundle $L_f = \mathbb{C}^F_V \times_{N_\mathbb{C}} \mathbb{C}$ over M^{toric}, where $t \in N_\mathbb{C}$ acts on $\mathbb{C}^F_V \times \mathbb{C}$ by sending $(z,\,u)$ to $(t \cdot z,\,t_f\,u)$. It follows that $c_f = \mathrm{c}(L_f)$. The holomorphic sections of L_f correspond to the holomorphic functions $s : \mathbb{C}^F_V \to \mathbb{C}$ which are equivariant in the sense that $s(t \cdot z) = t_f\,s(z)$ for every $t \in N_\mathbb{C}$ and $z \in \mathbb{C}^F_V$. It follows that the restriction to $\mathbb{C}^F_V$ of the linear form ϵ_f defines a holomorphic section of L_f over M^{toric}, which we also denote by ϵ_f. The zeroset of ϵ_f is equal to

$$S_f := {\mu_T}^{-1}(f),$$

the smooth complex codimension one toric subvariety of M^{toric} which in each $\mathbb{C}^{F_v}$ chart is determined by the equation $z_f = 0$. Because the zeros of ϵ_f are simple, $S_f = \mathrm{Div}(\epsilon_f)$, and $[L_f] = \delta(\mathrm{Div}(\epsilon_f)) = \delta(S_f)$, where $[L_f] \in \mathrm{H}^1(M^{\mathrm{toric}}, \mathcal{O}^\times)$ is the equivalence class of holomorphic line bundles containing L_f, and $\delta : \mathrm{H}^1(M^{\mathrm{toric}}, \mathcal{O}^\times) \to \mathrm{H}^2(M^{\mathrm{toric}}, \mathbb{Z})$ is the coboundary operator in the long exact sequence induced by the short exact sequence $0 \to \mathbb{Z} \to \mathcal{O} \xrightarrow{\mathrm{e}^{2\pi\,\mathrm{i}}} 1$. Here $\mathcal{O}$ and $\mathcal{O}^\times$ respectively denote the sheafs of germs of holomorphic and nowhere vanishing holomorphic functions on M^{toric}, and $\mathrm{e}^{2\pi\,\mathrm{i}}$ is the homomorphism $f \mapsto \mathrm{e}^{2\pi\,\mathrm{i}\,f}$ from the sheaf $\mathcal{O}$ of additive groups to the sheaf $\mathcal{O}^\times$ of multiplicative groups. It follows that

$$c_f = \mathrm{c}([L_f]) = \mathrm{c}(\delta(S_f)) = \mathrm{i}([S_f]), \tag{6.3}$$

where $\mathrm{i} : \mathrm{H}_{2n-2}(M^{\mathrm{toric}}, \mathbb{Z}) \xrightarrow{\sim} \mathrm{H}^2(M^{\mathrm{toric}}, \mathbb{Z})$ denotes the Poincaré duality isomorphism defined by the intersection numbers of chains. See Griffiths and Harris [**7**, bottom of p. 55 and Ch. 1, Sec. 1] for for the background theory.

Let Φ denote the collection of $\varphi \subset F$ such that $\varphi \subset F_v$ for some $v \in V$. Then the

$$\mathrm{face}_\varphi := \bigcap_{f \in \varphi} f, \quad \varphi \in \Phi,$$

are the closed faces of Δ of arbitrary dimensions. For each $\varphi \in \Phi$ we have the smooth toric subvariety

$$S_\varphi = {\mu_T}^{-1}(\mathrm{face}_\varphi)$$

of M^{toric}, which in each $\mathbb{C}^{F_v}$ chart is determined by the equations $z_f = 0$, $f \in \varphi$. It follows from Danilov [**5**, Cor. 7.4, Prop. 10.3 and 10.4, and Th. 10.8] that $\mathrm{H}^i(M^{\mathrm{toric}}, \mathcal{O}) = 0$ for every $i > 0$, $\mathrm{H}_*(M^{\mathrm{toric}}, \mathbb{Z})$ is generated by the homology classes of the S_φ, $\varphi \in \Phi$, and $\mathrm{H}_*(M^{\mathrm{toric}}, \mathbb{Z})$ has no torsion. Because $\mathrm{H}^2(M^{\mathrm{toric}}, \mathcal{O}) = 0$, every $c \in \mathrm{H}^2(M^{\mathrm{toric}}, \mathbb{Z})$ is equal to the Chern class of a holomorphic line bundle L over M^{toric}, which moreover is unique up to isomorphisms because $\mathrm{H}^1(M^{\mathrm{toric}}, \mathcal{O}) = 0$. Because $\mathrm{H}_{2n-2}(M^{\mathrm{toric}}, \mathbb{Z}) \simeq \mathrm{H}^2(M^{\mathrm{toric}}, \mathbb{Z})$ is generated by the $[S_f]$, $f \in F$, we have

$$\mathrm{c}(L) = c = \sum_{f \in F} n_f \ \mathrm{i}([S_f]) = \sum_{f \in F} n_f \ \mathrm{c}(L_f) = \mathrm{c}(\prod_{f \in F} L^{n_f}), \quad \text{hence} \quad L \simeq \prod_{f \in F} {L_f}^{n_f}$$

for suitable integers n_f, $f \in F$.

Let Φ_1 denote the set of all $\varphi \in \Phi$ such that $\dim_{\mathbb{R}}(S_\varphi) = 2$, that is, $\dim_{\mathbb{R}}(\mathrm{face}_\varphi) = 1$, or equivalently S_φ is a complex projective line = Riemann sphere embedded in M^{toric}. Because $\mathrm{H}^2(M^{\mathrm{toric}}, \mathbb{Z}) \simeq \mathrm{H}_{2n-2}(M^{\mathrm{toric}}, \mathbb{Z})$ has no torsion, the canonical homomorphism $\iota_{\text{de Rham}} : \mathrm{H}^2(M^{\mathrm{toric}}, \mathbb{Z}) \to \mathrm{H}^2_{\text{de Rham}}(M^{\mathrm{toric}})$ is injective, and $\gamma \in \mathrm{H}^2_{\text{de Rham}}(M^{\mathrm{toric}})$ belongs to the image if and only if $\int_{S_\varphi} \gamma \in \mathbb{Z}$ for every $\varphi \in \Phi_1$.

It follows in view of (6.2) that

$$[\sigma_\lambda^{\mathrm{toric}}] \in \iota_{\text{de Rham}}(\mathrm{H}^2(M^{\mathrm{toric}}, \mathbb{Z})) \quad \Leftrightarrow \quad \sum_{f \in F} \lambda_f \, (S_f \cdot S_\varphi) \in \mathbb{Z} \quad \forall \varphi \in \Phi_1, \tag{6.4}$$

where $S_f \cdot S_\varphi \in \mathbb{Z}$ denotes the intersection number of the toric subvarieties S_f and S_φ of M^{toric}. Furthermore, if $[\sigma_\lambda^{\mathrm{toric}}] \in \iota_{\text{de Rham}}(\mathrm{H}^2(M^{\mathrm{toric}}, \mathbb{Z}))$, then $[\sigma_\lambda^{\mathrm{toric}}] = \iota_{\text{de Rham}}(\mathrm{c}(L))$ for a holomorphic complex line bundle L over M^{toric}, which is uniquely determined up to isomorphisms and isomorphic to a product of integral powers of the L_f, $f \in F$.

For example, if Δ is a simplex, when M^{toric} is isomorphic to the n-dimensional complex projective space $\mathbb{CP}^n$, then the S_f are complex projective hyperplanes. All complex projective hyperplanes in $\mathbb{CP}^n$ define one and the same holomorphic line bundle, called the hyperplane bundle, the Chern class c of which generates $\mathrm{H}^2(\mathbb{CP}^n, \mathbb{Z}) \simeq \mathbb{Z}$. In this case $[\sigma_\lambda^{\mathrm{toric}}] = (\sum_{f \in F} \lambda_f) \, \iota_{\text{de Rham}}(c)$.

The nondegeneracy of $\sigma_\lambda^{\mathrm{toric}}$ implies that the Hermitian form h on M^{toric}, of which $\sigma_\lambda^{\mathrm{toric}}$ is the imaginary part, is nondegenerate at every point. At the origin in the $\mathbb{C}^{F_v}$-coordinates, $\sigma_\lambda^{\mathrm{toric}}$ is equal to a positive multiple of (3.4) and h is positive definite there. Because M^{toric} is connected, the signature of h is constant, hence h is positive definite everywhere. That is, $\sigma_\lambda^{\mathrm{toric}}$ is a *Kähler form* on M^{toric}.

Let σ be an arbitrary Kähler form, a closed two-form on M^{toric} equal to the imaginary part of a positive definite Hermitian structure h on M^{toric}. Because T is connected, the pullback $t^*(\sigma)$ of σ by $t \in T$ is homotopic and therefore cohomologous to σ. The average $\overline{h}$ of the positive definite Hermitian forms $t^*(h)$ over all $t \in T$ is positive definite, and the imaginary part of $\overline{h}$ is equal to the average $\overline{\sigma}$ of $\mathrm{Im}(t^*(h)) = t^*(\mathrm{Im}(h)) = t^*(\sigma)$ over all $t \in T$. It follows that $\overline{\sigma}$ is a T-invariant Kähler form with the same cohomology class as σ.

Because M^{toric} is simply connected, see Lemma 5·2, we have $\mathrm{H}^1_{\mathrm{de\ Rham}}(M^{\mathrm{toric}}) = 0$, hence the T-action is Hamiltonian with respect to $\overline{\sigma}$, with a momentum map $\mu : M^{\mathrm{toric}} \to \mathfrak{t}^*$. It follows from Delzant [**3**, Lemme 2.2 and (*) on p. 323] that each fiber of μ is a single T-orbit, $\Delta = \mu(M^{\mathrm{toric}})$ is a Delzant polytope, and the pre-images under μ of the relative interiors of the faces of Δ are the connected components of the orbit types for the T-action in M^{toric}. In particular the codimension one faces and the vertices of Δ correspond bijectively to the $f \in F$ and $v \in V$ used in the gluing construction of M^{toric}. For any $f \in F$, let Y_f denote the infinitesimal action of $X_f \in \mathfrak{t}$ on M^{toric}. Because $\overline{\sigma}$ is a Kähler form, we have in the complement of the zeroset of Y_f that $0 < \overline{\sigma}(Y_f,\, J\,Y_f) = -\,\mathrm{d}\langle X_f,\, \mu\rangle(JY_f)$, where J denotes the complex structure. If $v \in V$, $f \in F_v$, and we write the $\mathbb{C}^{F_v}$-coordinates as $z_f = x_f + \mathrm{i}\, y_f$ with $x_f,\, y_f \in \mathbb{R}$, then $2\pi\, Y_f = -y_f\, \partial/\partial x_f + x_f\, \partial/\partial y_f$, and therefore minus its J -image is equal to $x_f\, \partial/\partial x_f + y_f\, \partial/\partial y_f$. It follows that in the radial direction $x_f\, \partial/\partial x_f + y_f\, \partial/\partial y_f$ the function $\langle X_f,\, \mu\rangle$ is increasing close to $z_f = 0$, hence Δ is contained in the half space $\langle X_f,\, \xi\rangle + \lambda_f \geq 0$ if $\langle X_f,\, \xi\rangle + \lambda_f = 0$ on the face of Δ corresponding to f. Therefore there exists a $\lambda \in \Lambda$ such that Δ is equal to the polytope Δ_λ defined by (2.1).

Because $\mu(M^{\mathrm{toric}}) = \Delta = \Delta_\lambda = \mu_\lambda(M^{\mathrm{toric}})$, it follows from Delzant [**3**, Th. 2.1] that there exists a T-equivariant diffeomorphism ϑ of M^{toric} such that $\overline{\sigma} = \vartheta^*(\sigma_\lambda^{\mathrm{toric}})$ and $\mu = \mu_\lambda \circ \vartheta$. As the fibers of the momentum mappings are the T-orbits, ϑ is both T-equivariant and preserves the T-orbits. Because $\mu^{-1}(\Delta^{\mathrm{int}})$ is the set on which the action of T is free, it follows that for every $\xi \in \Delta^{\mathrm{int}}$ there is a unique $t = \tau(\xi) \in T$, such that $\vartheta(m) = t \cdot m$ for every $m \in \mu^{-1}(\{\xi\})$. A straightforward analysis of the equation $\vartheta(m) = \tau(\mu(m)) \cdot m$ shows that the function $\mu : \Delta^{\mathrm{int}} \to T$ extends to a smooth T-valued function on Δ, which we also denote by τ. Because Δ is simply connected, τ has a lift to the covering $\mathfrak{t}$ of T, that is, there exists a smooth mapping $\upsilon : \Delta \to \mathfrak{t}$ such that $\tau(\xi) = \exp(\upsilon(\xi))$ for every $\xi \in \Delta$. The mappings $\vartheta_s : m \to \exp(s\, \upsilon(\mu(m))) \cdot m$ form a smooth homotopy of diffeomorphisms, where $\vartheta_1 = \vartheta$ and $\vartheta_0 = 1$, the identity in M^{toric}. Therefore the action of ϑ on the de Rham cohomology groups is trivial, and we conclude that $[\sigma] = [\overline{\sigma}] = \vartheta^*([\sigma_\lambda^{\mathrm{toric}}]) = [\sigma_\lambda^{\mathrm{toric}}]$. In view of (6.2) and (6.3) this leads to the identity

$$\{\, [\sigma] \ \mid\ \sigma \text{ is a Kähler form on } M^{\mathrm{toric}} \,\} = \{\, \sum_{f \in F} \lambda_f \ \ \mathrm{i}([S_f]) \ \mid\ \lambda \in \Lambda \,\} \tag{6.5}$$

between subsets of $\mathrm{H}^2_{\mathrm{de\ Rham}}(M^{\mathrm{toric}})$. The set on the left and right hand side of (6.5) is equal to the open convex cone in Delzant [**3**, Prop. 3.2].

As averaging of a symplectic form does not necessarily yield a symplectic form, we do not make a statement about the cohomology classes of arbitrary symplectic forms on M^{toric}. We also note that if σ is a T-invariant symplectic form on M^{toric}, then it is a Kähler form on M^{toric} with respect to some complex structure on M^{toric}, which in general is very different from the one we started out with. For instance, negative $\sigma_\lambda^{\mathrm{toric}}$ is not a Kähler form with respect to the initial complex structure on M^{toric}.

7. *Examples*

7·1. *The complex projective space*

Let the Delzant polytope Δ be an n-dimensional simplex in $\mathfrak{t}^*$. That is, there are numberings f_i and v_i, $0 \leq i \leq n$, of F and V, respectively, such that $F_{v_j} = \{f_i \mid i \neq j\}$ for each j. Write $e_i = X_{f_i}$. Then the e_i, $1 \leq i \leq n$, form a $\mathbb{Z}$-basis of the integral lattice

$\mathfrak{t}_{\mathbb{Z}}$ in $\mathfrak{t}$, and there are $m_i \in \mathbb{Z}$ such that $e_0 = \sum_{i=1}^{n} m_i\, e_i$. For each $1 \leq j \leq n$ the vectors e_i such that $i \neq j$ form a $\mathbb{Z}$-basis of $\mathfrak{t}_{\mathbb{Z}}$, which condition is equivalent to $m_j = \pm 1$. The fan property implies that the intersection of the cone spanned by this basis with the cone spanned by the $e_1, \ldots, e_n$ is contained in a hyperplane, hence $m_j \leq 0$. The conclusion is that

$$e_0 = -\sum_{i=1}^{n} e_i. \tag{7.1}$$

In the sequel we identify f_i with its number i, whence $\mathbb{C}^F = \{(z_0, \ldots, z_n) \mid z_i \in \mathbb{C}\}$. The Delzant simplex (2.1), determined by the inequalities $\langle e_i, \xi \rangle + \lambda_i \geq 0$, $0 \leq i \leq n$, has a non-empty interior if and only if

$$\gamma := \sum_{i=0}^{n} \lambda_i > 0. \tag{7.2}$$

In the sequel we take $v = v_0$, that is, $\langle e_i, v \rangle + \lambda_i = 0$ for all $1 \leq i \leq n$. If we write $\xi_i = \langle e_i, \xi \rangle$, $1 \leq i \leq n$, when $\xi \in \mathfrak{t}^*$, then (3.2) yields that

$$\mu_v(z^v)_i = |z_i|^2/2 - \lambda_i, \quad 1 \leq i \leq n.$$

It follows from (3.7) that

$$r_0(\xi) = (2(-\sum_{i=1}^{n} \xi_i + \lambda_0))^{1/2},$$

and therefore (5.10) yields that

$$\zeta_i^v = z_i^v\, (2\gamma - \|z^v\|^2)^{-1/2}, \quad 1 \leq i \leq n, \tag{7.3}$$

where we have written

$$\|z^v\|^2 = \sum_{i=1}^{n} |z_i^v|^2.$$

Note that U_v is the open ball in $\mathbb{C}^n$ with center at the origin and radius equal to $(2c)^{1/2}$.

The equations (7.3) imply that

$$\|\zeta^v\|^2 = \|z_v\|^2/(2\gamma - \|z^v\|^2),$$

hence

$$\|z^v\|^2 = 2\gamma\, \|\zeta^v\|^2/(1 + \|\zeta^v\|^2).$$

Therefore the mapping $\theta^{v\,-1} : \zeta^v \mapsto z^v$ is given by the explicit formulas

$$z_i^v = \zeta_i^v\, (2\gamma/(1 + \|\zeta^v\|^2))^{1/2}, \quad 1 \leq i \leq n. \tag{7.4}$$

We have $\mathbb{C}_V^F = \mathbb{C}^{n+1} \setminus \{0\}$, $\mathfrak{n}_{\mathbb{C}}$ is the set of all $t \in \mathbb{C}^{n+1}$ such that $t_i = t_0$ for every $1 \leq i \leq n$, and we recognize the toric variety as the quotient of $\mathbb{C}^{n+1} \setminus \{0\}$ by the action of multiplications by nonzero complex scalars. That is, the toric variety is the n-dimensional complex projective space $\mathbb{CP}^n$. The symplectic form $\sigma_\lambda^{\text{toric}} = ({\theta_v}^{-1})^*(\sigma_v)$, where σ_v is the standard symplectic form (3.4), is equal to γ times the Fubini-Study form as defined in Griffiths and Harris [**7**, p. 30, 31]. As we have see in Section 6, the de Rham cohomology class of $\sigma_\lambda^{\text{toric}}$ is equal to γ times the de Rham cohomology image of the Chern class of the hyperplane bundle. This corresponds to the classically known fact that integral of the

Fubini-Study form over any complex projective line in $\mathbb{CP}^n$ is equal to one, see Griffiths and Harris [**7**, p. 122].

7·2. *The Hirzebruch surface*

Let $n = 2$. Then $k := \#(F) \geq 3$ and the faces form a cycle f_i, $i \in \mathbb{Z}/k\,\mathbb{Z}$, with vertices $v_i \in f_{i-1} \cap f_i$, when $F_{v_i} = \{f_{i-1}, f_i\}$. Write $e_i = X_{f_i}$. Then $e_{i+1} = b_i\, e_{i-1} + d_i\, e_i$ for b_i, $d_i \in \mathbb{Z}$. Because the matrix

$$M_i = \begin{pmatrix} 0 & b_i \\ 1 & d_i \end{pmatrix}$$

has an integral inverse, we have $b_i = \pm 1$. The fan property implies that the cone spanned by e_{i-1} and e_i intersects the cone spanned by e_i and e_{i+1} in $\mathbb{R}_{\geq 0}\, e_i$, hence $b_i \leq 0$, and therefore $b_i = -1$. The cocycle condition is equivalent to the condition that $M_k\, M_{k-1}\, \ldots\, M_2\, M_1 = 1$.

For $k = 3$ the cocycle condition leads to $d_1 = d_2 = d_3 = -1$, and we recover the complex projective plane as in Subsection 7·1. For $k = 4$, the case discussed in this subsection, the cocycle condition is equivalent to the equations $1 - d_2\, d_3 = 1$, $d_1 + (1 - d_1\, d_2)\, d_3 = 0$, $d_2 + (1 - d_2\, d_3)\, d_4 = 0$, and $1 - d_1\, d_2 - (d_1 + (1 - d_1\, d_2)\, d_3)\, d_4 = 1$. The solutions to these equations are $d_1 = d_3 = 0$, $d_4 = -d_2$, or $d_2 = d_4 = 0$, $d_3 = -d_1$, when Δ is a parallelogram or a trapezium. By means of a cyclic shift and/or an inversion in the numbering of the faces we can arrange that $e_3 = -e_1 + m\, e_2$ and $e_4 = -e_2$ for some $m \in \mathbb{Z}_{\geq 0}$, and we recognize the toric variety, obtained by gluing four copies of $\mathbb{C}^2$ together by means of the Laurent monomial coordinate transformations (5.7), as the *Hirzebruch surface* Σ_m, see Hirzebruch [**10**].

The Delzant polytope (2.1) is determined by the inequalities $\langle e_i,\, \xi\rangle + \lambda_i \geq 0$, $1 \leq i \leq 4$, which is a quadrangle if and only if

$$\gamma_\pm := \lambda_1 + \lambda_3 \pm m\, \lambda_4 > 0, \tag{7.5}$$

and the inequalities imply that $\lambda_2 + \lambda_4 = \gamma_+ + \gamma_- > 0$.

In the sequel we take for v the vertex determined by the equations $\langle e_i,\, \xi\rangle + \lambda_i = 0$ for $i = 1, 2$, where $F_v = \{1, 2\}$. If we write $\xi_i = \langle e_i,\, \xi\rangle$, $1 \leq i \leq 2$, when $\xi \in \mathfrak{t}^*$, then (3.2) yields that

$$\mu_v(z^v)_i = |z_i|^2/2 - \lambda_i, \quad 1 \leq i \leq 2.$$

It follows from (3.7) that

$$r_3(\xi) = (2(-\xi_1 + m\, \xi_2 + \lambda_3))^{1/2}, \quad r_4(\xi) = (2(-\xi_2 + \lambda_4))^{1/2}$$

and therefore (5.10) yields that

$$\zeta_1^v = z_1^v\, (2\gamma_- |z_1^v|^2 + m\, |z_2^v|^2)^{-1/2}, \tag{7.6}$$

$$\zeta_2^v = z_2^v\, (2\gamma_- |z_1^v|^2 + m\, |z_2^v|^2)^{m/2}\, (2(\gamma_+ + \gamma_-) - |z_2^v|^2)^{-1/2}. \tag{7.7}$$

If we write $t_i = |z_i^v|^2$ and $\tau_i = |\zeta_i^v|^2$, then this leads to the equations

$$\tau_1 = t_1/(2\gamma_- - t_1 + m\, t_2),$$
$$\tau_2 = t_2\, (2\gamma_- + t_1 + m\, t_2)^m/(2(\gamma_+ + \gamma_-) - t_2)$$

for t_1, t_2. If we solve t_1 from the first equation,

$$t_1 = (2\gamma_- + m\, t_2)\, \tau_1/(1 + \tau_1),$$

and substitute this into the second equation, then this leads to the polynomial equation

$$(1+\tau_1)^m \, \tau_2 \, (2(\gamma_+ + \gamma_-) - t_2) = t_2 \, (2\gamma_- + m \, t_2)^m \tag{7.8}$$

of degree $m+1$ for t_2. If we substract the left hand side from the right hand side then the derivative with respect to t_2 is strictly positive, and one readily obtains that for every $\tau_1, \tau_2 \in \mathbb{R}_{\geq 0}$ there is a unique solution $t_2 \in \mathbb{R}_{\geq 0}$, confirming the first statement in Lemma 5·5.

On the other hand, if we work over $\mathbb{C}$, and view both the parameter $\varepsilon := (1+\tau_1)^m \, \tau_2$ and the unknown t_2 as elements of the complex projective line $\mathbb{CP}^1$, then the equation (7.8) defines a complex algebraic curve C in the (t_2, ε)-plane $\mathbb{CP}^1 \times \mathbb{CP}^1$, where the restriction to C of the projection to the first variable t_2 is a complex analytic diffeomorphism from C onto $\mathbb{CP}^1$, as on C we have that ε is a complex analytic function of t_2. In particular C is irreducible. The restriction to C of the projection to the second variable ε is an $(m+1)$-fold branched covering. Over $\varepsilon = 0$ and over $\varepsilon = \infty$ we have that m of the $m+1$ branches come together, whereas there are two more branch points on the ε-line over which only two of the branches come together. The fact that C is irreducible implies that the part of C over the complement of the branch points is connected, and therefore the analytic continuation of any solution t_2 of (7.8), as a complex analytic function of ε in the complement of the branch points, will reach each and every other branch if ε runs over a suitable loop. In other words, the solution t_2 is an algebraic function of ε of degree $m+1$, and no branch of a solution is of lower degree. This holds in particular for our solutions $t_2 \in \mathbb{R}_{\geq 0}$ for $\varepsilon \in \mathbb{R}_{\geq 0}$.

REFERENCES

[**1**] R. Abraham and J.E. Marsden: *Foundations of Mechanics*. Benjamin/Cummings Publ. Co., London, etc., 1978.

[**2**] M. Audin: *The Topology of Torus Actions on Symplectic Manifolds*. Birkhäuser, Basel, Boston, Berlin, 1991.

[**3**] T. Delzant: Hamiltoniens périodiques et images convexes de l'application moment. *Bull. Soc. Math. France* **116** (1988) 315–339.

[**4**] M. Demazure: Sous-groupes algébriques de rang maximum du groupe de Cremona. *Ann. scient. Éc. Norm. Sup.* **3** (1970) 507–588.

[**5**] V.I. Danilov: The geometry of toric varieties. *Russ. Math. Surveys* **33**:2 (1978) 97–154, translated from *Uspekhi Mat. Nauk SSSR* **33**:2 (1978) 85–134.

[**6**] J.J. Duistermaat and G.J. Heckman: On the variation in the cohomology of the symplectic form of the reduced phase space. *Invent. math.* **69** (1982) 259–268.

[**7**] P. Griffiths and J. Harris: *Principles of Algebraic Geometry*. J. Wiley & Sons, Inc., New York, etc., 1978.

[**8**] V. Guillemin: *Moment Maps and Combinatorial Invariants of Hamiltonian* T^n*-spaces*. Birkhäuser, Boston, etc., 1994.

[**9**] V. Guillemin: Kaehler structures on toric varieties. *J. Differential Geometry* **40** (1994) 285–309.

[**10**] F. Hirzebruch: Über eine Klasse von einfach-zusammenhĝenden komplexen Mannigfaltigkeiten. *Mathematische Annalen* **124** (1951) 77–86.

[**11**] A. Pelayo: Topology of spaces of equivariant symplectic embeddings. *Proc. Amer. Math. Soc.* **135** (2007) 277–288.

[**12**] R.T. Rockafellar: *Convex Analysis*. Princeton University Press, princeton, N.J., 1970.